GRAPH ALGEBRAS, EXEL-LACA ALGEBRAS, AND ULTRAGRAPH ALGEBRAS COINCIDE UP TO MORITA EQUIVALENCE

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ABSTRACT. We prove that the classes of graph algebras, Exel-Laca algebras, and ultragraph algebras coincide up to Morita equivalence. This result answers the long-standing open question of whether every Exel-Laca algebra is Morita equivalent to a graph algebra. Given an ultragraph $\mathcal{G}$ we construct a directed graph E such that $C^*(\mathcal{G})$ is isomorphic to a full corner of $C^*(E)$. As applications, we characterize real rank zero for ultragraph algebras and describe quotients of ultragraph algebras by gauge-invariant ideals.

1. INTRODUCTION

In 1980 Cuntz and Krieger introduced a class of C^*-algebras associated to finite matrices [4]. Specifically, if A is an $n \times n$ $\{0, 1\}$-matrix with no zero rows, then the Cuntz-Krieger algebra $\mathcal{O}_A$ is generated by partial isometries $S_1, \ldots, S_n$ such that $S_i^* S_i = \sum_{A(i,j)=1} S_j S_j^*$. Shortly thereafter Enomoto, Fujii, and Watatani [5, 9, 23] observed that Cuntz and Krieger's algebras could be described very naturally in terms of finite directed graphs. Given a finite directed graph E in which every vertex emits at least one edge, the corresponding C^*-algebra $C^*(E)$ is generated by mutually orthogonal projections P_v associated to the vertices and partial isometries S_e associated to the edges such that $S_e^* S_e = P_{r(e)}$ and $P_v = \sum_{s(e)=v} S_e S_e^*$, where $r(e)$ and $s(e)$ denote the range and source of an edge e.

Attempting to generalize the theory of Cuntz-Krieger algebras to countably infinite generating sets resulted in two very prominent classes of C^*-algebras: graph C^*-algebras and Exel-Laca algebras. To motivate our results we briefly describe each of these classes. The key issue for both generalizations is that infinite sums of projections, which a naive approach would suggest, cannot converge in norm.

To generalize graph C^*-algebras to infinite graphs, the key modification is to require the relation $P_v = \sum_{s(e)=v} S_e S_e^*$ to hold only when the sum is finite and nonempty. This theory has been explored extensively in many papers (see [12, 13, 2, 8] for seminal results, and [14]

for a survey). Graph C^*-algebras include many C^*-algebras besides the Cuntz-Krieger algebras; in particular, graph C^*-algebras include the Toeplitz algebra, continuous functions on the circle, all finite-dimensional C^*-algebras, many AF-algebras, many purely infinite simple C^*-algebras, and many Type I C^*-algebras. From a representation-theoretic point of view, the class of graph C^*-algebras is broader still: every AF-algebra is Morita equivalent to a graph C^*-algebra, and any Kirchberg algebra with free K_1-group is Morita equivalent to a graph C^*-algebra.

The approach taken for Exel-Laca algebras is to allow the matrix A to be infinite. Here rows containing infinitely many nonzero entries lead to an infinite sum of projections, which does not give a sensible relation. However, Exel and Laca observed that even when rows of the matrix contain infinitely many nonzero entries, formal combinations of the Cuntz-Krieger relations can result in relations of the form $\prod_{i \in X} S_i^* S_i \prod_{j \in Y} (1 - S_j^* S_j) = \sum_{k \in Z} S_k S_k^*$, where X, Y, and Z are all finite. It is precisely these finite relations that are imposed in the definition of the Exel-Laca algebra. As with the graph C^*-algebras, the Exel-Laca algebras include many classes of C^*-algebras in addition to the Cuntz-Krieger algebras, and numerous authors have studied their structure [6, 7, 15, 17].

Without too much effort, one can show that neither the class of graph C^*-algebras nor the class of Exel-Laca algebras is a subclass of the other. Specifically, there exist graph C^*-algebras that are not isomorphic to any Exel-Laca algebra [19, Proposition A.16.2], and there exist Exel-Laca algebras that are not isomorphic to any graph C^*-algebra [15, Example 4.2 and Remark 4.4]. This shows, in particular, that there is merit in studying both classes and that results for one class are not special cases of results for the other. It also begs the question, "How different are the classes of graph C^*-algebras and Exel-Laca algebras?" Although each contains different isomorphism classes of C^*-algebras, a natural follow-up question is to ask about Morita equivalence. Specifically,

QUESTION 1: Is every graph C^*-algebra Morita equivalent to an Exel-Laca algebra?

QUESTION 2: Is every Exel-Laca algebra Morita equivalent to a graph C^*-algebra?

While the question of isomorphism is easy to sort out, the Morita equivalence questions posed above are much more difficult. Question 1 was answered in the affirmative by Fowler, Laca, and Raeburn in [8]. In particular, if $C^*(E)$ is a graph C^*-algebra, then one may form a graph $\tilde{E}$ with no sinks or sources, by adding tails to the sinks of E and heads to the sources of E. A standard argument shows that $C^*(\tilde{E})$ is Morita equivalent to $C^*(E)$ (see [2, Lemma 1.2], for example), and Fowler, Laca, and Raeburn proved that the C^*-algebra of a graph with no sinks and no sources is isomorphic to an Exel-Laca algebra [8, Theorem 10].

On the other hand, Question 2 has remained an open problem for nearly a decade. The various invariants calculated for graph C^*-algebras and Exel-Laca algebras have not been able to discern any Exel-Laca algebras that are not Morita equivalent to a graph C^*-algebra. For example, the attainable K-theories for both classes are the same: all

countable free abelian groups arise as K_1-groups together with all countable abelian groups as K_0-groups (see [18] and [7]). Nevertheless, up to this point there has been no method for constructing a graph E from a matrix A so that $C^*(E)$ is Morita equivalent to $\mathcal{O}_A$.

In this paper we provide an affirmative answer to Question 2 using a generalization of a graph known as an ultragraph. Ultragraphs and the associated C^*-algebras were introduced by the fourth author to unify the study of graph C^*-algebras and Exel-Laca algebras [20, 21]. An ultragraph is a generalization of a graph in which the range of an edge is a (possibly infinite) set of vertices, rather than a single vertex. The ultragraph C^*-algebra is then determined by generators satisfying relations very similar to those for graph C^*-algebras (see Section 2). The fourth author has shown that every graph algebra is isomorphic to an ultragraph algebra [20, Proposition 3.1], every Exel-Laca algebra is isomorphic to an ultragraph algebra [20, Theorem 4.5, Remark 4.6], and moreover there are ultragraph algebras that are not isomorphic to any graph algebra and are not isomorphic to any Exel-Laca algebra [21, §5]. Thus the class of ultragraph algebras is strictly larger than the union of the two classes of Exel-Laca algebras and of graph algebras.

In addition to providing a framework for studying graph algebras and Exel-Laca algebras simultaneously, ultragraph algebras also give an alternate viewpoint for studying Exel-Laca algebras. In particular, if A is a (possibly infinite) $\{0,1\}$-matrix, and if we let $\mathcal{G}$ be ultragraph with edge matrix A, then the Exel-Laca algebra $\mathcal{O}_A$ is isomorphic to the ultragraph algebra $C^*(\mathcal{G})$. In much of the seminal work done on Exel-Laca algebras [6, 7, 17], the structure of the C^*-algebra $\mathcal{O}_A$ is related to properties of the infinite matrix A (see [6, §2–§10] and also [7, Definition 4.1 and Theorem 4.5]) as well as properties of an associated graph $\mathrm{Gr}(A)$ with edge matrix A (see [6, Definition 10.5, Theorem 13.1, Theorem 14.1, and Theorem 16.2] and [17, Theorem 8]). Unfortunately, these correspondences are often of limited use since the properties of the matrix can be difficult to visualize, and the graph $\mathrm{Gr}(A)$ does not entirely reflect the structure of the Exel-Laca algebra $\mathcal{O}_A$ (see [21, Example 3.14 and Example 3.15]). Another approach is to represent properties of the Exel-Laca algebra in terms of the ultragraph $\mathcal{G}_A$ [20, 21]. This is a useful technique because it gives an additional way to look at properties of Exel-Laca algebras, the ultragraph $\mathcal{G}_A$ reflects much of the fine structure of the Exel-Laca algebra $\mathcal{O}_A$, and furthermore the interplay between the ultragraph and the associated C^*-algebra has a visual nature similar to what occurs with graphs and graph C^*-algebras.

In this paper we prove that the classes of graph algebras, Exel-Laca algebras, and ultragraph algebras coincide up to Morita equivalence. This provides an affirmative answer to Question 2 above, and additionally shows that no new Morita equivalence classes are obtained in the strictly larger class of ultragraph algebras. Given an ultragraph $\mathcal{G}$ we build a graph E with the property that $C^*(\mathcal{G})$ is isomorphic to a full corner of $C^*(E)$. Combined with other known results, this shows our three classes of C^*-algebras coincide up to Morita equivalence. Since our construction is concrete, we are also able to use graph algebra results to analyze the structure of ultragraph algebras. In particular,

we characterize real rank zero for ultragraph algebras, and describe the quotients of ultragraph algebras by gauge-invariant ideals. Of course, these structure results also give corresponding results for Exel-Laca algebras as special cases, and these results are new as well. In addition, our construction implicitly gives a method for taking a $\{0,1\}$-matrix A and forming a graph E with the property that the Exel-Laca algebra is isomorphic to a full corner of the graph C^*-algebra $C^*(E)$ (see Remark 5.24).

It is interesting to note that we have used ultragraphs to answer Question 2, even though this question is intrinsically only about graph C^*-algebras and Exel-Laca algebras. Indeed it is difficult to see how to answer Question 2 without at least implicit recourse to ultragraphs. This provides additional evidence that ultragraphs are a useful and natural tool for exploring the relationship between Exel-Laca algebras and graph C^*-algebras.

This paper is organized as follows. After some preliminaries in Section 2, we describe our construction in Section 3 and explain how to build a graph E from an ultragraph $\mathcal{G}$. Since this construction is somewhat involved, we also provide a detailed example for a particular ultragraph at the end of this section. In Section 4 we analyze the path structure of the graph constructed by our method. In Section 5 we show that there is an isomorphism ϕ from the ultragraph algebra $C^*(\mathcal{G})$ to a full corner $PC^*(E)P$ of $C^*(E)$, and we use this result to show that an ultragraph algebra $C^*(\mathcal{G})$ has real rank zero if and only if $\mathcal{G}$ satisfies Condition (K). In Section 6, we prove that the induced bijection $I \mapsto C^*(E)\phi(I)C^*(E)$ restricts to a bijection between gauge-invariant ideals of $C^*(\mathcal{G})$ and gauge-invariant ideals of $C^*(E)$. In Section 7, we give a complete description of the gauge-invariant ideal structure of ultragraph algebras commenced in [11], by describing the quotient of an ultragraph algebra by a gauge-invariant ideal.

2. Preliminaries

For a set X, let $\mathcal{P}(X)$ denote the collection of all subsets of X. We recall from [20] the definitions of an ultragraph and of a Cuntz-Krieger family for an ultragraph.

Definition 2.1. ([20, Definition 2.1]) An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s\colon \mathcal{G}^1 \to G^0$ and $r\colon \mathcal{G}^1 \to \mathcal{P}(G^0) \setminus \{\emptyset\}$.

The original definition of a Cuntz-Krieger family for an ultragraph $\mathcal{G}$ appears as [20, Definition 2.7]. However, for our purposes, it will be more convenient to work with the Exel-Laca $\mathcal{G}$-families of [11, Definition 3.3]. To give this definition, we first recall that for finite subsets λ and μ of $\mathcal{G}^1$, we define

$$r(\lambda, \mu) := \bigcap_{e \in \lambda} r(e) \setminus \bigcup_{f \in \mu} r(f) \in \mathcal{P}(G^0).$$

Definition 2.2. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. A collection of projections $\{p_v : v \in G^0\}$ and $\{q_e : e \in \mathcal{G}^1\}$ is said to satisfy *Condition (EL)* if the following hold:

    (1) the elements of $\{p_v : v \in G^0\}$ are pairwise orthogonal,

    (2) the elements of $\{q_e : e \in \mathcal{G}^1\}$ pairwise commute,

    (3) $p_v q_e = p_v$ if $v \in r(e)$, and $p_v q_e = 0$ if $v \notin r(e)$,

    (4) $\prod_{e \in \lambda} q_e \prod_{f \in \mu} (1 - q_f) = \sum_{v \in r(\lambda, \mu)} p_v$ for all finite subsets λ, μ of $\mathcal{G}^1$ such that $\lambda \cap \mu = \emptyset$, $\lambda \neq \emptyset$ and $r(\lambda, \mu)$ is finite.

Given an ultragraph $\mathcal{G}$, we write G^0_{rg} for the set $\{v \in G^0 : s^{-1}(v)$ is finite and nonempty$\}$ of *regular vertices* of $\mathcal{G}$.

Definition 2.3. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, an *Exel-Laca $\mathcal{G}$-family* is a collection of projections $\{p_v : v \in G^0\}$ and partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal final projections for which

    (1) the collection $\{p_v : v \in G^0\} \cup \{s_e^* s_e : e \in \mathcal{G}^1\}$ satisfies Condition (EL),

    (2) $s_e s_e^* \leq p_{s(e)}$ for all $e \in \mathcal{G}^1$,

    (3) $p_v = \sum_{s(e)=v} s_e s_e^*$ for $v \in G^0_{\mathrm{rg}}$.

Our use of the notation $\mathcal{G}^0$ in what follows will also be in keeping with [11] rather than with [20, 21].

By a *lattice* in $\mathcal{P}(X)$, we mean a collection of subsets of X which is closed under finite intersections and unions. By an *algebra* in $\mathcal{P}(X)$, we mean a lattice in $\mathcal{P}(X)$ which is closed under taking relative complements. As in [11], we denote by $\mathcal{G}^0$ the smallest algebra in $\mathcal{P}(G^0)$ which contains both $\{\{v\} : v \in G^0\}$ and $\{r(e) : e \in \mathcal{G}^1\}$ (by contrast, in [20, 21], $\mathcal{G}^0$ denotes the smallest lattice in $\mathcal{P}(G^0)$ containing these sets). A *representation* of an algebra $\mathfrak{A}$ in a C^*-algebra B is a collection $\{p_A : a \in \mathfrak{A}\}$ of mutually commuting projections in B such that $p_{A \cap B} = p_A p_B$, $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ and $p_{A \setminus B} = p_A - p_{A \cap B}$ for all $A, B \in \mathfrak{A}$.

Given an Exel-Laca $\mathcal{G}$-family $\{p_v : v \in G^0\}$, $\{s_e : e \in \mathcal{G}^1\}$ in a C^*-algebra B, [11, Proposition 3.4] shows that there is a unique representation $\{p_A : A \in \mathcal{G}^0\}$ of $\mathcal{G}^0$ such that $p_{r(e)} = s_e^* s_e$ for all $e \in E^0$, and $p_{\{v\}} = p_v$ for all $v \in G^0$. In particular, given an Exel-Laca $\mathcal{G}$-family $\{p_v, s_e\}$, we will without comment denote the resulting representation of $\mathcal{G}^0$ by $\{p_A : A \in \mathcal{G}^0\}$; in particular, $p_{r(e)}$ denotes $s_e^* s_e$, and $p_{\{v\}}$ and p_v are one and the same.

3. A DIRECTED GRAPH CONSTRUCTED FROM AN ULTRAGRAPH

The purpose of this section is to construct a graph $E = (E^0, E^1, r_E, s_E)$ from an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$. Our construction involves a choice of a listing of $\mathcal{G}^1$ and of a function σ with certain properties described in Lemma 3.7; in particular, different choices of listings and of sigma will yield different graphs. In Section 5, we will prove that the ultragraph algebra $C^*(\mathcal{G})$ is isomorphic to a full corner of the graph algebra $C^*(E)$, regardless of the choices made.

Notation 3.1. Fix $n \in \mathbb{N} = \{1, 2, \ldots\}$, and $\omega \in \{0, 1\}^n$. For $i = 1, 2, \ldots, n$, we denote by $\omega_i \in \{0, 1\}$ the ith coordinate of ω, and we denote n by $|\omega|$.

We express ω as $(\omega_1, \omega_2, \ldots, \omega_n)$. We define $(\omega, 0), (\omega, 1) \in \{0,1\}^{n+1}$ by $(\omega, 0) := (\omega_1, \omega_2, \ldots, \omega_n, 0)$ and $(\omega, 1) := (\omega_1, \omega_2, \ldots, \omega_n, 1)$. For $m \in \mathbb{N}$ with $m \leq n$, we define $\omega|_m \in \{0,1\}^m$ by $\omega|_m = (\omega_1, \omega_2, \ldots, \omega_m)$. The elements $(0,0,\ldots,0,0)$ and $(0,0,\ldots,0,1)$ in $\{0,1\}^n$ are denoted by 0^n and $(0^{n-1}, 1)$.

Let $\mathcal{G}$ be an ultragraph $(G^0, \mathcal{G}^1, r, s)$. Fix an ordering on $\mathcal{G}^1 = \{e_1, e_2, e_3, \ldots\}$. (This list may be finite or countably infinite.) Using the same notation as established in [11, Section 2], we define $r(\omega) := \bigcap_{\omega_i=1} r(e_i) \setminus \bigcup_{\omega_j=0} r(e_j) \subset G^0$ for $\omega \in \{0,1\}^n \setminus \{0^n\}$, and $\Delta_n := \{\omega \in \{0,1\}^n \setminus \{0^n\} : |r(\omega)| = \infty\}$. If $\omega \in \{0,1\}^n$ and $i \in \{0,1\}$ so that $\omega' := (\omega, i) \in \{0,1\}^{n+1}$, we somewhat inaccurately write $r(\omega, i)$, rather than $r((\omega, i))$, for $r(\omega')$.

Definition 3.2. We define $\Delta := \bigsqcup_{n=1}^{\infty} \Delta_n$. So for $\omega \in \Delta$, we have $\omega \in \Delta_{|\omega|}$.

Remark 3.3. Since $r(\omega) = r(\omega, 0) \sqcup r(\omega, 1)$ for each $\omega \in \Delta$, an element $\omega \in \{0,1\}^n \setminus \{0^n\}$ is in Δ if and only if at least one of the elements $(\omega, 0)$ and $(\omega, 1)$ is in Δ.

Definition 3.4. We define $\Gamma_0 := \{(0^n, 1) : n \geq 0, |r(0^n, 1)| = \infty\} \subset \Delta$, and $\Gamma_+ := \Delta \setminus \Gamma_0$.

We point out that $(0^n, 1)$ means (1) when $n = 0$. Also, by Remark 3.3, if $n \in \mathbb{N}$, $\omega \in \Delta_{n+1}$, and $\omega|_n \neq 0^n$, then $\omega|_n \in \Delta_n$. Since $\omega \in \Delta_{n+1}$ satisfies $\omega|_n = 0^n$ if and only if $\omega = (0^n, 1) \in \Gamma_0$, it follows that $\Gamma_+ = \{\omega \in \Delta : |\omega| > 1 \text{ and } \omega|_{|\omega|-1} \in \Delta\}$.

Definition 3.5. Let $W_+ := \bigcup_{\omega \in \Delta} r(\omega) \subset G^0$, and $W_0 := G^0 \setminus W_+$.

Lemma 3.6. *We have $W_+ = \bigsqcup_{\omega \in \Gamma_0} r(\omega)$.*

Proof. For $\omega \in \Delta$, let $m(\omega) := \min\{k : \omega_k = 1\}$. Then $\omega|_{m(\omega)} \in \Gamma_0$ and $r(\omega) \subset r(\omega|_{m(\omega)})$. Thus $W_+ = \bigcup_{\omega \in \Gamma_0} r(\omega)$. Finally, the sets $r(\omega)$ and $r(\omega')$ are disjoint for distinct $\omega, \omega' \in \Gamma_0$ by definition. $\square$

Lemma 3.7. *There exists a function $\sigma \colon W_+ \to \Delta$ such that $v \in r(\sigma(v))$ for each $v \in W_+$, and such that $\sigma^{-1}(\omega)$ is finite (possibly empty) for each $\omega \in \Delta$.*

Proof. Let
$$W_\infty := \{v \in W_+ : v \in r(\omega) \text{ for infinitely many } \omega \in \Delta\}.$$
For $v \in W_+ \setminus W_\infty$, we define $\sigma(v)$ to be the element of Δ with $v \in r(\sigma(v))$ for which $|\sigma(v)|$ is maximal. Fix an ordering $\{v_1, v_2, v_3, \ldots\}$ of W_∞, and fix $k \in \mathbb{N}$. The definition of W_∞ implies that the set $N_k := \{n \geq k : v_k \in r(\omega) \text{ for some } \omega \in \Delta_n\}$ is infinite. Let n denote the minimal element of N_k. By Lemma 3.6, there is a unique $\omega \in \Delta_n$ such that $v_k \in r(\omega)$. We define $\sigma(v_k) := \omega$.

By definition, we have $v \in r(\sigma(v))$ for each $v \in W_+$. Fix $\omega \in \Delta$. We must show that $\sigma^{-1}(\omega)$ is finite. The set $\sigma^{-1}(\omega) \cap W_\infty$ is finite because it is a subset of $\{v_1, v_2, \ldots, v_{|\omega|}\} \subset W_\infty$. The set $\sigma^{-1}(\omega) \cap (W_+ \setminus W_\infty)$ is empty if both $(\omega, 0)$ and $(\omega, 1)$ are in Δ. Otherwise the set $\sigma^{-1}(\omega) \cap (W_+ \setminus W_\infty)$ coincides with the finite set $r(\omega, i)$ for $i = 0$ or 1. Thus $\sigma^{-1}(\omega)$ is finite. $\square$

Definition 3.8. Fix a function $\sigma \colon W_+ \to \Delta$ as in Lemma 3.7. Extend σ to a function $\sigma \colon G^0 \to \Delta \cup \{\emptyset\}$ by setting $\sigma(v) = \emptyset$ for $v \in W_0$.

We take the convention that $|\emptyset| = 0$ so that $v \mapsto |\sigma(v)|$ is a function from G^0 to the nonnegative integers. In particular, $v \in W_0$ if and only if $|\sigma(v)| = 0$, and $v \in W_+$ if and only if $|\sigma(v)| \geq 1$.

Definition 3.9. For each $n \in \mathbb{N}$, we define a subset $X(e_n)$ of $G^0 \sqcup \Delta$ by

$$X(e_n) := \big\{ v \in r(e_n) : |\sigma(v)| < n \big\} \sqcup \big\{ \omega \in \Delta_n : \omega_n = 1 \big\}.$$

Remark 3.10. The occurrence of the symbol e in the notation $X(e_n)$ is redundant; we might just as well label this set $X(n)$ or X_n. However, we feel that it is helpful to give some hint that the role of the n in this notation is to pick out an edge e_n from our chosen listing of $\mathcal{G}^1$.

Lemma 3.11. *For each $n \in \mathbb{N}$, the set $X(e_n)$ is nonempty and finite.*

Proof. Fix $n \in \mathbb{N}$. To see that $X(e_n)$ is nonempty, suppose that $\{v \in r(e_n) : |\sigma(v)| < n\} = \emptyset$. Since $r(e_n) \neq \emptyset$, there exists $v \in r(e_n)$ such that $|\sigma(v)| \geq n$. Set $\omega = \sigma(v)|_n$. Since $v \in r(\sigma(v)) \subset r(\omega)$, we have $\omega_n = 1$ and $|r(\omega)| = \infty$, so $\omega \in X(e_n)$. Thus $X(e_n)$ is nonempty.

For $v \in r(e_n)$ with $|\sigma(v)| = 0$, the element $\omega \in \{0, 1\}^n$ satisfying $v \in r(\omega)$ is not in Δ_n by definition. Hence the set $\{v \in r(e_n) : |\sigma(v)| = 0\}$ is finite, because it is a subset of the union of finitely many finite sets $r(\omega)$ where $\omega \in \{0, 1\}^n \setminus \Delta_n$. Since $\sigma^{-1}(\omega)$ is finite for all $\omega \in \Delta$ and since $\{\omega \in \Delta : |\omega| < n\}$ is finite, we have $|\{v \in r(e_n) : 0 < |\sigma(v)| < n\}| < \infty$. Since Δ_n is finite, $\{\omega \in \Delta_n : \omega_n = 1\}$ is also finite, and thus $X(e_n)$ is finite. $\qquad\square$

Definition 3.12. We define a graph $E = (E^0, E^1, r_E, s_E)$ as follows:

$$E^0 := G^0 \sqcup \Delta,$$
$$E^1 := \{\overline{x} : x \in W_+ \sqcup \Gamma_+\} \sqcup \big\{(e_n, x) : e_n \in \mathcal{G}^1, \ x \in X(e_n)\big\},$$
$$r_E(\overline{x}) := x, \quad r_E((e_n, x)) := x,$$
$$s_E(\overline{v}) := \sigma(v), \quad s_E(\overline{\omega}) := \omega|_{|\omega|-1}, \quad s_E((e_n, x)) := s(e_n).$$

Remark 3.13. Just as in Remark 3.10, the symbol e here is redundant; we could simply have denoted the edge (e_n, x) by (n, x). We have chosen notation which is suggestive of the fact that the n is specifying an element of $\mathcal{G}^1$ via our chosen listing.

For the following proposition, recall E^0_{rg} denotes the set $\{v \in E^0 : s_E^{-1}(v)$ is finite and nonempty$\}$ of regular vertices of E. Also recall from Section 2 that G^0_{rg} denotes the set of regular vertices of $\mathcal{G}$.

Proposition 3.14. *We have $E^0_{\mathrm{rg}} = G^0_{\mathrm{rg}} \sqcup \Delta$.*

Proof. For $v \in G^0$, we have $s_E^{-1}(v) = \bigsqcup_{s(e_n)=v} \{(e_n, x) : x \in X(e_n)\}$. Hence Lemma 3.11 implies that $s_E^{-1}(v)$ is nonempty and finite if and only if $s^{-1}(v) \subset \mathcal{G}^1$ is nonempty and finite. Hence $v \in E^0_{\mathrm{rg}}$ if and only if $v \in G^0_{\mathrm{rg}}$. Thus $E^0_{\mathrm{rg}} \cap G^0 = G^0_{\mathrm{rg}}$.

Fix $\omega \in \Delta$. By Remark 3.3, we have $(\omega, i) \in \Delta$ for at least one of $i = 0$ and $i = 1$. Since $s_E\big(\overline{(\omega, i)}\big) = \omega$, the set $s_E^{-1}(\omega)$ is not empty. By definition of σ, the set $s_E^{-1}(\omega)$ is finite. Hence $\omega \in E_{\mathrm{rg}}^0$. Thus $E_{\mathrm{rg}}^0 = G_{\mathrm{rg}}^0 \sqcup \Delta$. $\square$

An example of the graph constructed from an ultragraph. We present an example of our construction, including an illustration. This concrete example will, we hope, help the reader to visualise the general construction and to keep track of notation. The definition of the ultragraph $\mathcal{G}$ in this example may seem involved, but has been chosen to illustrate as many of the features of our construction as possible with a diagram containing relatively few vertices.

Example 3.15. For the duration of this example we will omit the parentheses and commas when describing elements of Δ. For example, the element $(0, 0, 1) \in \{0, 1\}^3$ will be denoted 001.

We define an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ as follows. Let $G^0 := \{v_n : n \in \mathbb{N}\}$ and $\mathcal{G}^1 := \{e_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $s(e_n) := v_n$. For $k \in \mathbb{N}$, let

$$r(e_{2k-1}) := \{v_m : (k+2) \text{ divides } m\},$$

$$r(e_{2k}) := \{v_m : m \leq k^2 \text{ and } 4 \text{ does not divide } m\}.$$

We will construct a graph E from $\mathcal{G}$ as described in the earlier part of this section. We have displayed $\mathcal{G}$ and E in Figure 1. To construct E, we must first choose a function $\sigma : W_+ \to \Delta$ as in Lemma 3.7. To do this, we first describe Δ and W_+.

Fix $n \geq 1$ and $\omega \in \{0, 1\}^n \setminus \{0^n\}$. If $\omega_{2k} = 1$ for some $k \in \mathbb{N}$, then $r(\omega) \subset \{v_1, v_2, \ldots, v_{k^2}\}$ is finite, so $\omega \notin \Delta$. If $\omega_{2k-1} = 0$ and $\omega_{2l-1} = 1$ for $k, l \in \mathbb{N}$ with $(k+2) \mid (l+2)$, then $r(\omega) = \emptyset$, so $\omega \notin \Delta$. Indeed, we have $\omega \in \Delta$ if and only if:

(1) $\omega_i = 1$ for some i;
(2) $\omega_i = 0$ for all even i; and
(3) whenever k satisfies $\omega_{2k-1} = 0$, we have $(k+2) \nmid \mathrm{lcm}\{l+2 : \omega_{2l-1} = 1\}$.

For example, note that $\omega = 1010100$ and $\omega = 1010000$ are not in Δ (see Figure 1) because these have $\omega_1 = \omega_3 = 1$, but $\omega_7 = 0$. Similarly, elements of the form $0****1$ are missing.

To describe Γ_0, first observe that an element of the form 0^n1 can belong to Δ only if n is even. For an element ω of the form $0^{2n}1$, conditions (1) and (2) above are trivially satisfied, and $\{l+2 : \omega_{2l-1} = 1\} = \{n+3\}$, so condition (3) holds if and only if $k+2 \nmid n+3$ for all k such that $2k - 1 < 2n + 1$; that is, if and only if $n + 3$ is equal to 4 or is an odd prime number. Thus

$$\Gamma_0 = \big\{0^{2(p-3)}1 : p = 4 \text{ or } p \text{ is an odd prime number}\big\}$$
$$= \{1, 001, 00001, 0^81, 0^{16}1, 0^{20}1, 0^{28}1, \ldots\}.$$

We now describe W_+. By Lemma 3.6, W_+ is the disjoint union of the sets $r(0^{2(p-3)}1)$ where p runs through 4 and all odd prime numbers. We have

$$r(1) = \{v_m : 3 \mid m\},$$
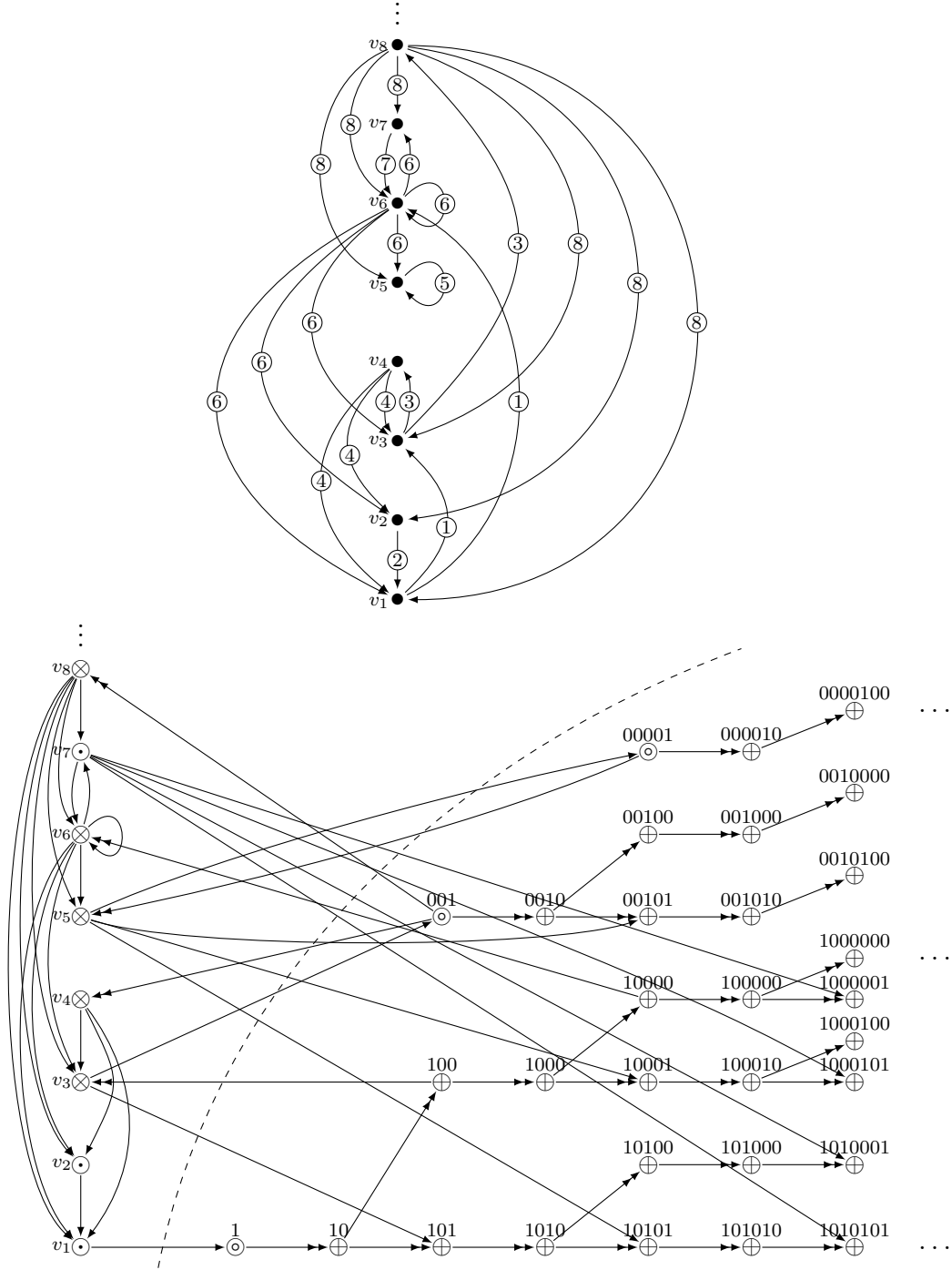
FIGURE 1. The ultragraph $\mathcal{G}$ (top) and graph E (bottom) of Example 3.15

$$r(001) = \{v_m : 3 \nmid m \text{ and } 4 \mid m\},$$
$$r(00001) = \{v_m : 3 \nmid m, \ 4 \nmid m, \text{ and } 5 \mid m\},$$

and for an odd prime number p greater than 5,

$$r(0^{2(p-3)}1) = \{v_m : 3 \nmid m, \ 4 \nmid m, \ldots, (p-1) \nmid m, \ p \mid m, \text{ and } m > (p-3)^2\}.$$

This implies that $v_1, v_2 \notin W_+$ and that $v_m \in W_+$ whenever $3 \mid m$, $4 \mid m$, or $5 \mid m$. Fix $m \in \mathbb{N} \setminus (\{1, 2\} \cup 3\mathbb{N} \cup 4\mathbb{N} \cup 5\mathbb{N})$. Let p be the smallest odd prime divisor of m. Then p is greater than 5. Moreover $v_m \in W_+$ if and only if $v_m \in r(0^{2(p-3)}1)$, which is equivalent to $m > (p-3)^2$. Let $k = m/p \in \mathbb{N}$. Since p is the smallest odd prime divisor of m, either $k = 1$, $k = 2$, or $k \geq p$. If $k = 1$ or $k = 2$, we have $m = kp \leq (p-3)^2$ and hence $v_m \notin W_+$. If $k \geq p$, then $m = kp \geq p^2 > (p-3)^2$, so $v_m \in W_+$. Recall that $W^0 = G^0 \setminus W_+$. We have proved that W_0 may be described as

$$W_0 = \{v_p, v_{2p} : p = 1 \text{ or } p \text{ is an odd prime number greater than } 5\}$$
$$= \{v_1, v_2, v_7, v_{11}, v_{13}, v_{14}, v_{17}, v_{19}, v_{22}, v_{23}, \ldots\},$$

and then W_+ is the complement of this set:

$$W_+ = G^0 \setminus W_0 = \{v_3, v_4, v_5, v_6, v_8, v_9, v_{10}, v_{12}, v_{15}, v_{16}, v_{18}, v_{20}, v_{21}, \ldots\}.$$

We now define a function $\sigma \colon W_+ \to \Delta$ with the properties described in Lemma 3.7. Since each $r(e_{2k}) = \{v_n : n \leq k^2, 4 \nmid n\}$, the set $W_\infty \subset W_+$ described in the proof of Lemma 3.7 is $\{v_m \in W_+ : 4 \mid m\}$. Thus $\{v_4, v_8, v_{12}, v_{16}, \ldots\}$ is an ordering of W_∞. For $k \in \mathbb{N}$, let $n := \max\{3, k\}$. Then n is the smallest integer such that $n \geq k$ and $v_{4k} \in \bigsqcup_{\omega \in \Delta_n} r(\omega)$. Define $\sigma(v_{4k})$ to be the unique element ω of Δ_n such that $v_{4k} \in r(\omega)$. So

$$\sigma(v_4) = \sigma(v_8) = 001, \quad \sigma(v_{12}) = 101, \quad \sigma(v_{16}) = 0010, \quad \sigma(v_{20}) = 00101, \quad \ldots$$

For $v_m \in W_+ \setminus W_\infty$, let k be the minimal integer such that $m \leq k^2$. Then $n := 2k - 1$ is the maximal integer such that $v_m \in \bigsqcup_{\omega \in \Delta_n} r(\omega)$. We define $\sigma(v_m)$ to be the unique element ω of Δ_n such that $v_m \in r(\omega)$. So

$$\sigma(v_3) = 100, \quad \sigma(v_5) = 00001, \quad \sigma(v_6) = 10000, \quad \sigma(v_9) = 10000,$$
$$\sigma(v_{10}) = 0000100, \quad \sigma(v_{15}) = 1000100, \quad \sigma(v_{18}) = 100000100, \quad \ldots$$

By our convention that $\sigma(v) = \emptyset$ whenever $v \in W_0$, we have

$$\emptyset = \sigma(v_1) = \sigma(v_2) = \sigma(v_7) = \sigma(v_{11}) = \sigma(v_{13}) = \sigma(v_{14}) = \sigma(v_{17}) = \sigma(v_{19}) = \cdots.$$

We also have

$$X(e_1) = \{1\}, \quad X(e_2) = \{v_1\}, \quad X(e_3) = \{001, 101\}, \quad X(e_4) = \{v_1, v_2, v_3\},$$
$$X(e_5) = \{00001, 00101, 10001, 10101\}, \quad X(e_6) = \{v_1, v_2, v_3, v_5, v_6, v_7, v_9\},$$
$$X(e_7) = \{v_6, v_{12}, v_{24}, 1000001, 1000101, 1010001, 1010101\},$$
$$X(e_8) = \{v_1, v_2, v_3, v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{13}, v_{14}, v_{15}\}, \quad \ldots$$

This is all the information required to draw E, and we have done so in Figure 1. To distinguish the various special sets of vertices discussed above, we draw vertices using four different symbols as follows: vertices of the form $\odot$ belong to W_0; those of the form $\otimes$ belong to W_+; those of the form $\circledcirc$ belong to Γ_0; and those of the form $\oplus$ belong to Γ_+. The dashed arc separates G^0 on the left from Δ on the right.

Edges drawn as double-headed arrows are of the form $\overline{x}$ where $x \in W_+ \sqcup \Gamma_+$, and edges drawn as single-headed arrows are of the form (e_n, x) where $e_n \in \mathcal{G}^1$ and $x \in X(e_n)$. Since $s_E(\overline{x}) \in \Delta$ and $r_E(\overline{x}) = x$ for all $x \in W_+ \sqcup \Gamma_+$, and since $s_E((e_n, x)) = s(e_n) = v_n$ and $r_E((e_n, x)) = x$ for all n and $x \in X(e_n)$, once we know the type of an edge, the edge is uniquely determined by its source and its range. Thus it is not necessary to label the edges in the figure.

4. Paths and Condition (K)

For this section, we fix an ultragraph $\mathcal{G}$, and make a choice of an ordering $\{e_1, e_2, \dots\}$ of $\mathcal{G}^1$ and a function σ as in Lemma 3.7. Let $E = (E^0, E^1, r_E, s_E)$ be the graph constructed from $\mathcal{G}$ as in Definition 3.12. We relate the path structure of E to that of $\mathcal{G}$. In particular, we show that $\mathcal{G}$ satisfies Condition (K) as in [11] if and only if E satisfies Condition (K) as in [13]. Condition (K) was introduced in [13] to characterise those graphs in whose C^*-algebras every ideal is gauge-invariant. In Section 6, we will combine our results in this section with our main result Theorem 5.22 to deduce from Kumjian, Pask, Raeburn and Renault's result the corresponding theorem for ultragraph C^*-algebras.

We recall some terminology for graphs (see, for example, [12, 2]; note that our edge direction convention agrees with that used in these papers and in [11], which is opposite to that used in [14]).

For each integer $n \geq 2$, we write

$$E^n := \{\alpha = \alpha_1 \alpha_2 \cdots \alpha_n : \alpha_i \in E^1 \text{ and } s_E(\alpha_{i+1}) = r_E(\alpha_i) \text{ for all } i\},$$

and $E^* := \bigsqcup_{n=0}^{\infty} E^n$. The elements of E^* are called *paths*. The *length* of a path α is the integer $|\alpha|$ such that $\alpha \in E^{|\alpha|}$. We extend the range and source maps to E^* as follows. For $v \in E^0$, we write $r_E(v) = s_E(v) = v$. For $\alpha \in E^* \setminus E^0$ we write $r_E(\alpha) = r_E(\alpha_{|\alpha|})$ and $s_E(\alpha) = s_E(\alpha_1)$.

For $\alpha, \beta \in E^*$ such that $r_E(\alpha) = s_E(\beta)$, we may form the path $\alpha\beta \in E^*$ by concatenation. Thus E^* becomes a category whose unit space is $E^0 \subset E^*$. For a vertex $v \in E^0$, a *return path based at v* is a path α of nonzero length with $s_E(\alpha) = r_E(\alpha) = v$. A return path $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ based at v is called a *first-return path* if $s_E(\alpha_i) \neq v$ for $i = 2, 3, \dots, n$. We say that E satisfies *Condition (K)* if no vertex is the base of exactly one first-return path (equivalently, each vertex is either the base of no return path, or is the base of at least two first-return paths).

Consider the subgraph F of E with the same vertices as E, and edges $F^1 = \{\overline{x} : x \in W_+ \sqcup \Gamma_+\} \subset E^1$. Note that $F^1 \subset E^1$ is the set of all edges in E^1 starting from $\Delta \subset E^0$.

In Figure 1 the elements of F^1 are the double-headed arrows. Then

$$F^* = E^0 \sqcup \left\{ \overline{x}_1 \overline{x}_2 \cdots \overline{x}_n \in E^* : n \in \mathbb{N}, x_i \in W_+ \sqcup \Gamma_+ \right\} \subset E^*.$$

In particular $\alpha \in E^*$ belongs to F^* if and only if it contains no edges of the form (e_n, x) where $e_n \in \mathcal{G}^1$ and $x \in X(e_n)$.

Lemma 4.1. *Every $\alpha \in E^*$ can be uniquely expressed as*

$$\alpha = g_0 \cdot (e_{n_1}, x_1) \cdot g_1 \cdot (e_{n_2}, x_2) \cdot g_2 \cdot \cdots \cdot (e_{n_k}, x_k) \cdot g_k$$

where each $e_{n_i} \in \mathcal{G}^1$, $x_i \in X(e_{n_i})$ and $g_i \in F^*$.

Proof. Let $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$. Whenever α_i and α_{i+1} are both of the form (e_n, x), rewrite $\alpha_i \alpha_{i+1} = \alpha_i r_E(\alpha_i) \alpha_{i+1}$ (recall that $r_E(\alpha_i) \in E^0$ belongs to F^* by definition). Now by grouping sequences of consecutive edges from F^1, we obtain an expression for α of the desired form. This expression is clearly unique. □

In the graph F, we can distinguish the sets W_0, W_+, Γ_0, and Γ_+ as follows.

- An element in W_0 emits no edges, and receives no edges.
- An element in W_+ emits no edges, and receives exactly one edge.
- An element in Γ_0 emits a finite and nonzero number of edges, and receives no edges.
- An element in Γ_+ emits a finite and nonzero number of edges, and receives exactly one edges.

Next we describe the paths of the graph F. To do so, the following notation is useful.

Definition 4.2. For $n \in \mathbb{N}$ and $\omega \in \{0,1\}^n \setminus \{0^n\}$ we set

$$r'(\omega) := \{v \in r(\omega) : |\sigma(v)| \geq n\} = \{v \in G^0 : |\sigma(v)| \geq n, \ \sigma(v)|_n = \omega\}.$$

To see that the two sets in the definition above coincide, it suffices to see that for $v \in G^0$ with $|\sigma(v)| \geq n$, we have $v \in r(\omega)$ if and only if $\sigma(v)|_n = \omega$. For this, observe that $v \in r(\sigma(v)) \subset r(\sigma(v)|_n)$ and that there is at most one $\omega \in \{0,1\}^n \setminus \{0^n\}$ such that $v \in r(\omega)$.

Lemma 4.3. *For $\omega \notin \Delta$ the set $r'(\omega)$ is empty.*

Proof. If $\omega \in \{0,1\}^n \setminus \{0^n\}$ for some $n \in \mathbb{N}$, and suppose that $r'(\omega)$ is not empty. Then there exists $v \in G^0$ with $|\sigma(v)| \geq n$ and $\sigma(v)|_n = \omega$. Since $\sigma(v) \in \Delta$ and $\omega \neq 0^n$, we have $\omega \in \Delta$. This shows $r'(\omega) = \emptyset$ for $\omega \notin \Delta$. □

Lemma 4.4. *For each $n \in \mathbb{N}$, we have $r(e_n) = (X(e_n) \cap G^0) \sqcup \left(\bigsqcup_{\omega \in X(e_n) \cap \Delta} r'(\omega) \right)$.*

Proof. We have $r(e_n) = \bigsqcup_{\omega \in \{0,1\}^n, \, \omega_n = 1} r(\omega)$. The definition of $X(e_n)$ (see Definition 3.9) guarantees that $X(e_n) \cap G^0 = \{v \in r(e_n) : |\sigma(v)| < n\}$. For $\omega \in \{0,1\}^n$ with $\omega_n = 1$, we have $r'(\omega) = r(\omega) \setminus \{v \in r(\omega) : |\sigma(v)| < n\}$ by definition. Hence

$$r(e_n) = (X(e_n) \cap G^0) \sqcup \left(\bigsqcup_{\omega \in \{0,1\}^n, \, \omega_n = 1} r'(\omega) \right).$$

Finally $r'(\omega) = \emptyset$ for $\omega \notin \Delta$ by Lemma 4.3. □

Remark 4.5. For $\omega \in \{0,1\}^n \setminus \{0^n\}$ one can show $r'(\omega) = r'(\omega, 0) \sqcup r'(\omega, 1) \sqcup \sigma^{-1}(\omega)$, using the fact $\sigma^{-1}(\omega) = \{v \in r(\omega) : |\sigma(v)| = n\}$. We omit the routine proof because we do not use it, but we remark this fact because this relates to Lemma 5.9 (this can be proved using Lemma 4.6 (3) below).

Lemma 4.6. *The graph F contains no return paths, and each $\alpha \in F^*$ is uniquely determined by $s_E(\alpha)$ and $r_E(\alpha)$. Moreover,*

(1) *every path in $\alpha \in F$ of nonzero length satisfies $s_E(\alpha) \in \Delta$;*
(2) *there is a path in F from $\omega \in \Delta$ to $\omega' \in \Delta$ if and only if $|\omega| \le |\omega'|$ and $\omega = \omega'|_{|\omega|}$.*
(3) *there is a path in F from $\omega \in \Delta$ to $v \in G^0$ if and only if $v \in r'(\omega)$; and*

Proof. Fix $e \in F^1$. Then either $r_E(e) \in G^0$ and hence is a sink in F, or else $s_E(e) \in \Delta_n$ and $r_E(e) \in \Delta_{n+1}$ for some $n \in \mathbb{N}$. Thus F contains no return paths.

Now suppose that $\alpha, \alpha' \in F^*$ satisfy $r_E(\alpha) = r_E(\alpha')$ and $s_E(\alpha) = s_E(\alpha')$. Without loss of generality, we may assume that $|\alpha| \ge |\alpha'|$. By definition of F, each vertex $v \in E^0$ receives at most one edge in F^1, so $\alpha = \beta \alpha'$ for some $\beta \in F^*$. This forces $s_E(\beta) = s_E(\alpha) = s_E(\alpha') = r_E(\beta)$, and then β has length 0 by the preceding paragraph, and $\alpha = \alpha'$.

By definition of F, we have $s_E(F^1) = \Delta$, which proves (1). As explained in the first paragraph, a path α from Δ_n to $\omega' \in \Delta$ must have the form $\alpha = \overline{\omega'|_{n+1}} \cdot \overline{\omega'|_{n+2}} \cdots \overline{\omega'|_{|\omega'|-1}} \cdot \overline{\omega'}$. This expression makes sense if and only if $n \le |\omega'|$ and $\omega := \omega'|_n$ is in Δ_n, and then α has source ω. This proves (2). For (3), fix $\omega \in \Delta_n$ and $v \in G^0$. There is a path from ω to v if and only if $v \in W_+$ and there is a path from ω to $\sigma(v)$. By (2), this occurs if and only if $n \le |\sigma(v)|$ and $\sigma(v)|_n = \omega$ (in particular, $n = |\omega|$). Thus, there is a path from ω to v if and only if $v \in r'(\omega)$. $\qquad\square$

Definition 4.7. Lemma 4.6 implies that for each $x \in E^0$, there is a unique element $f_x \in F^*$ such that $r_E(f_x) = x$ and $s_E(f_x) \in W_0 \sqcup \Gamma_0$. Observe that

- For $x \in W_0 \sqcup \Gamma_0$, we have $f_x = x$.
- For $x = \omega \in \Gamma_+$, we have

$$f_x := \overline{\omega|_{m+1}} \cdot \overline{\omega|_{m+2}} \cdots \overline{\omega|_{|\omega|-1}} \cdot \overline{\omega}$$

  where $m = \min\{k : \omega_k = 1\}$.
- For $x = v \in W_+$, we have $f_x = f_{\sigma(v)}\overline{v}$.

Example 4.8. Consider the ultragraph of Example 3.15, and the corresponding graph E illustrated there.

- We have $f_{v_1} = v_1$ and $f_{00001} = 00001$ since $v_1 \in W_0$ and $00001 \in \Gamma_0$.
- We have $f_{001000} = \overline{0010} \cdot \overline{00100} \cdot \overline{001000}$.
- We have $f_{v_6} = \overline{10} \cdot \overline{100} \cdot \overline{1000} \cdot \overline{10000} \cdot \overline{v_6}$.

In the second two instances, it is easy to see that f_x is the unique path in double-headed arrows from Γ_0 (that is, a vertex of the form $\odot$) to x.

Lemma 4.9. *For fixed $v, w \in G^0$, the map*

$$(4.1) \qquad g_0 \cdot (e_{n_1}, x_1) \cdot g_1 \cdot (e_{n_2}, x_2) \cdot g_2 \cdots (e_{n_k}, x_k) \cdot g_k \mapsto \begin{cases} g_0 & \text{if } k = 0 \\ e_{n_1} e_{n_2} \cdots e_{n_k} & \text{otherwise} \end{cases}$$

is a bijection between paths in E from v to w and paths in $\mathcal{G}$ beginning at v whose ranges contain w where each $e_{n_i} \in \mathcal{G}^1$, $x_i \in X(e_{n_i})$ and $g_i \in F^*$ as in Lemma 4.1.

Proof. First note that we have $g_0 = v$ because $v \in G^0$ emits no edges in F. Since $s_E((e_{n_i}, x_i)) = s(e_{n_i}) \in G^0$, to show that the map is well-defined and bijection, it suffices to show that for each $e_n \in \mathcal{G}^1$ and $w \in G^0$ there exists a path $\alpha = (e_n, x) \cdot g$ where $x \in X(e_n)$ and $g \in F^*$ satisfying $r_E((e_n, x)) = w$ if and only if $w \in r(e_n)$, and in this case $x \in X(e_n)$ and $g \in F^*$ are unique. This follows from Lemma 4.4 and Lemma 4.6 (3). $\square$

We introduced notions of paths and Condition (K) for graphs at the beginning of this section. We now recall the corresponding notions for ultragraphs. A *path* in an ultragraph is a sequence $\alpha = \alpha_1 \alpha_2 \ldots \alpha_{|\alpha|}$ of edges such that $s(\alpha_{i+1}) \in r(\alpha_i)$ for all i. We write $s(\alpha) = s(\alpha_1)$ and $r(\alpha) = r(\alpha_{|\alpha|})$. A *return path* is a path α such that $s(\alpha) \in r(\alpha)$. A *first-return path* is a return path α such that $s(\alpha) \neq s(\alpha_i)$ for any $i \geq 1$. As in [11, Section 7], we say an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ satisfies *Condition (K)* if no vertex is the base of exactly one first-return path.

Proposition 4.10. *The graph E satisfies Condition (K) if and only if the ultragraph $\mathcal{G}$ satisfies Condition (K).*

Proof. Lemma 4.6 implies that every return path in E passes through some vertex in G^0. Hence E satisfies Condition (K) if and only if no vertex in G^0 is the base of exactly one first-return path in E. This in turn happens if and only if $\mathcal{G}$ satisfies Condition (K) by Lemma 4.9. $\square$

5. Full corners of graph algebras

Once again, we fix an ultragraph $\mathcal{G}$ and a graph E constructed from $\mathcal{G}$ as in Definition 3.12. We will show that the ultragraph algebra $C^*(\mathcal{G})$ is isomorphic to a full corner of the graph algebra $C^*(E)$.

Definition 5.1. The *graph algebra* $C^*(E)$ of the graph $E = (E^0, E^1, r_E, s_E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{q_x : x \in E^0\}$ and partial isometries $\{t_\alpha : \alpha \in E^1\}$ with mutually orthogonal ranges satisfying the Cuntz-Krieger relations:

(1) $t_\alpha^* t_\alpha = q_{r_E(\alpha)}$ for all $\alpha \in E^1$;
(2) $t_\alpha t_\alpha^* \leq q_{s_E(\alpha)}$ for all $\alpha \in E^1$; and
(3) $q_x = \sum_{s_E(\alpha) = x} t_\alpha t_\alpha^*$ for $x \in E_{\text{rg}}^0$.

As usual, for a path $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ in E we define $t_\alpha \in C^*(E)$ by $t_\alpha = t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n}$. For $x \in E^0 \subset E^*$, the notation t_x is understood as q_x. The properties (1) and (2) in Definition 5.1 hold for all $\alpha \in E^*$.

Definition 5.2. For each $x \in E^0$ define a partial isometry $U_x := t_{f_x} \in C^*(E)$ where $f_x \in F^*$ is as in Definition 4.7.

By definition of f_x and the Cuntz-Krieger relations, we have $U_x^* U_x = q_x$ and $U_x U_x^* \leq q_{s_E(f_x)}$ for $x \in E^0$.

Lemma 5.3. *For $x, y \in E^0$ with $x \neq y$, we have*

$$(U_x U_x^*)(U_y U_y^*) = \begin{cases} U_x U_x^* & \text{if there exists a path in } F \text{ from } x \text{ to } y, \\ U_y U_y^* & \text{if there exists a path in } F \text{ from } y \text{ to } x, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $n \in \mathbb{N}$ and $x, y \in X(e_n)$ with $x \neq y$, we have $U_x^* U_y = 0$.

Proof. Without loss of generality, we may assume $|f_x| \leq |f_y|$. Then $U_x^* U_y \neq 0$ if and only if f_y extends f_x, and in this case $(U_x U_x^*) U_y = U_y$. By the uniqueness of f_y in F^* stated in Definition 4.7, f_y extends f_x exactly when there exists a path in F from x to y.

For the last statement, observe that by Lemma 4.4 and Lemma 4.6, there exist no paths in F among vertices in $X(e_n)$. Hence $U_x^* U_y = U_x^* U_x U_x^* U_y U_y^* U_y = 0$. $\qquad\square$

Definition 5.4. For $v \in G^0$, we set $P_v := U_v U_v^*$. For $e_n \in \mathcal{G}^1$, we set

$$S_{e_n} := U_{s(e_n)} \sum_{x \in X(e_n)} t_{(e_n, x)} U_x^*.$$

It is clear that P_v is a nonzero projection, and the last statement of Lemma 5.3 implies that S_{e_n} is a partial isometry. We will show in Proposition 5.15 that the collection $\{P_v : v \in G^0\}$ and $\{S_{e_n} : e_n \in \mathcal{G}^1\}$ is an Exel-Laca $\mathcal{G}$-family in $C^*(E)$.

Definition 5.5. For $e_n \in \mathcal{G}^1$, we define $Q_{e_n} := S_{e_n}^* S_{e_n} \in C^*(E)$. For $\omega \in \bigsqcup_{n=1}^{\infty} (\{0, 1\}^n \setminus \{0^n\})$, we define

$$Q'_\omega := \begin{cases} U_\omega U_\omega^* & \text{if } \omega \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

The projections Q'_ω are related to the sets $r'(\omega)$ of the preceding section (see Proposition 5.17).

Lemma 5.6. *The collections $\{P_v : v \in G^0\}$ and $\{Q'_\omega : \omega \in \bigsqcup_{n=1}^{\infty} (\{0, 1\}^n \setminus \{0^n\})\}$ of projections satisfy the following:*

(1) *$\{P_v : v \in G^0\}$ are pairwise orthogonal.*
(2) *$\{Q'_\omega : \omega \in \bigsqcup_{n=1}^{\infty} (\{0, 1\}^n \setminus \{0^n\})\}$ pairwise commute.*
(3) *$\{Q'_\omega : \omega \in \{0, 1\}^n \setminus \{0^n\}\}$ are pairwise orthogonal for each $n \in \mathbb{N}$.*
(4) *$P_v Q'_\omega = Q'_\omega P_v = P_v$ if $v \in r'(\omega)$, and $P_v Q'_\omega = Q'_\omega P_v = 0$ if $v \notin r'(\omega)$.*

Proof. By Lemma 4.6 paths in F are uniquely determined by their ranges and sources, and Lemma 5.3 shows how the $U_x U_x^*$ multiply. The four assertions follow immediately. $\quad\square$

Lemma 5.7. *For $n \in \mathbb{N}$, we have*

$$Q_{e_n} = \sum_{x \in X(e_n)} U_x U_x^* = \sum_{\substack{\omega \in \{0,1\}^n \\ \omega_n = 1}} Q'_\omega + \sum_{\substack{v \in r(e_n) \\ |\sigma(v)| < n}} P_v.$$

Proof. We compute:

$$\begin{aligned}
Q_{e_n} &= S_{e_n}^* S_{e_n} \\
&= \left(\sum_{x \in X(e_n)} U_x t_{(e_n,x)}^* \right) U_{s(e_n)}^* U_{s(e_n)} \left(\sum_{y \in X(e_n)} t_{(e_n,y)} U_y^* \right) \\
&= \sum_{x,y \in X(e_n)} (U_x t_{(e_n,x)}^* t_{(e_n,y)} U_y^*).
\end{aligned}$$

Since $t_{(e_n,x)}^* t_{(e_n,y)} = 0$ for $x, y \in X(e_n)$ with $x \neq y$, we deduce that $Q_{e_n} = \sum_{x \in X(e_n)} U_x U_x^*$ as claimed.

By the definition of $X(e_n)$, we have

$$\sum_{x \in X(e_n)} U_x U_x^* = \sum_{\substack{\omega \in \{0,1\}^n \\ \omega_n = 1}} Q'_\omega + \sum_{\substack{v \in r(e_n) \\ |\sigma(v)| < n}} P_v. \qquad \square$$

Lemma 5.8. *The collection of projections $\{Q_e : e \in \mathcal{G}^1\}$ satisfy the following:*

(1) $\{Q_e : e \in \mathcal{G}^1\}$ *pairwise commute.*
(2) $P_v Q_e = Q_e P_v = P_v$ *if $v \in r(e)$, and $P_v Q_e = Q_e P_v = 0$ if $v \notin r(e)$.*
(3) *For $n \in \mathbb{N}$ and $\omega \in \{0,1\}^n \setminus \{0^n\}$, we have $Q'_\omega Q_{e_n} = Q_{e_n} Q'_\omega = Q'_\omega$ if $\omega_n = 1$, and $Q'_\omega Q_{e_n} = Q_{e_n} Q'_\omega = 0$ if $\omega_n = 0$.*

Proof. Assertions (1) and (2) follow from routine calculations using Lemma 5.6 and Lemma 5.7. Assertion (3) follows from similar calculations using the decomposition of $r(e_n)$ from Lemma 4.4. $\qquad \square$

Lemma 5.9. *For $\omega \in \{0,1\}^n \setminus \{0^n\}$, we have*

$$Q'_\omega = Q'_{(\omega,0)} + Q'_{(\omega,1)} + \sum_{\substack{v \in r(\omega) \\ |\sigma(v)| = n}} P_v.$$

Proof. For $\omega \notin \Delta$ both sides of the equation are zero. For $\omega \in \Delta$, we have $\omega \in E_{\mathrm{rg}}^0$ by Proposition 3.14. Hence by the Cuntz-Krieger relations, we have

$$\begin{aligned}
q_\omega &= \sum_{\substack{i \in \{0,1\} \\ (\omega,i) \in \Delta}} t_{\overline{(\omega,i)}} t_{\overline{(\omega,i)}}^* + \sum_{\substack{v \in G^0 \\ \sigma(v) = \omega}} t_{\overline{v}} t_{\overline{v}}^* \\
&= \sum_{\substack{i \in \{0,1\} \\ (\omega,i) \in \Delta}} t_{\overline{(\omega,i)}} t_{\overline{(\omega,i)}}^* + \sum_{\substack{v \in r(\omega) \\ |\sigma(v)| = n}} t_{\overline{v}} t_{\overline{v}}^*.
\end{aligned}$$

Multiplying by U_ω on the left and by U_ω^* on the right gives the desired equation. $\square$

Definition 5.10. For $n \in \mathbb{N}$ and $\omega \in \{0,1\}^n \setminus \{0^n\}$, we define $Q_\omega \in C^*(E)$ by

$$Q_\omega := \prod_{\omega_i=1} Q_{e_i} \prod_{\omega_j=0} (1 - Q_{e_j}).$$

Lemma 5.11. *For every* $\omega \in \{0,1\}^n \setminus \{0^n\}$, *we have*

$$(5.1) \qquad Q_\omega = Q_\omega' + \sum_{\substack{v \in r(\omega) \\ |\sigma(v)| < |\omega|}} P_v.$$

Proof. We proceed by induction on n. The case $n = 1$ follows from Lemma 5.7 because $Q_\omega = Q_{e_1}$ and $r(\omega) = r(e_1)$ for the only element $\omega = (1)$ of $\{0,1\}^1 \setminus \{0^1\}$.

Fix $n \in \mathbb{N}$, and suppose as an inductive hypothesis that Equation (5.1) holds for all elements of $\{0,1\}^n \setminus \{0^n\}$. Then for each $\theta \in \{0,1\}^n \setminus \{0^n\}$, the inductive hypothesis and Lemma 5.9 imply that

$$\begin{aligned} Q_\theta &= Q_\theta' + \sum_{\substack{v \in r(\theta) \\ |\sigma(v)| < n}} P_v \\ &= \left(Q_{(\theta,0)}' + Q_{(\theta,1)}' + \sum_{\substack{v \in r(\theta) \\ |\sigma(v)| = n}} P_v \right) + \sum_{\substack{v \in r(\theta) \\ |\sigma(v)| < n}} P_v \\ (5.2) \qquad &= Q_{(\theta,0)}' + Q_{(\theta,1)}' + \sum_{\substack{v \in r(\theta) \\ |\sigma(v)| < n+1}} P_v. \end{aligned}$$

Now fix $\omega \in \{0,1\}^{n+1} \setminus \{0^{n+1}\}$; we must establish Equation 5.1 for this ω. We consider three cases: $\omega = (\theta, 1)$ for some $\theta \in \{0,1\}^n \setminus \{0^n\}$; $\omega = (\theta, 0)$ for some $\theta \in \{0,1\}^n \setminus \{0^n\}$; or $\omega = (0^n, 1)$.

First suppose that $\omega = (\theta, 1)$. Then $Q_\omega = Q_{(\theta,1)} = Q_\theta Q_{e_{n+1}}$. Combining this with Lemma 5.8 (2) and (3) and with 5.2, we obtain

$$Q_\omega = Q_{(\theta,1)}' + \sum_{\substack{v \in r(\theta) \cap r(e_{n+1}) \\ |\sigma(v)| < n+1}} P_v = Q_{(\theta,1)}' + \sum_{\substack{v \in r(\theta,1) \\ |\sigma(v)| < n+1}} P_v.$$

Now suppose that $\omega = (\theta, 0)$. We may apply the conclusion of the preceding paragraph to $(\theta, 1)$ for calculate

$$Q_\omega = Q_{(\theta,0)} = Q_\theta - Q_{(\theta,1)} = Q_{(\theta,0)}' + \sum_{\substack{v \in r(\theta) \setminus r(\theta,1) \\ |\sigma(v)| < n+1}} P_v = Q_{(\theta,0)}' + \sum_{\substack{v \in r(\theta,0) \\ |\sigma(v)| < n+1}} P_v.$$

Finally, suppose that $\omega = (0^n, 1)$. Then we may apply the conclusion of the preceding paragraph to each $Q'_{(\theta,1)}$ where $\theta \in \{0,1\}^n \setminus \{0^n\}$ to calculate

$$Q_\omega = Q_{(0^n,1)}$$

$$= Q_{e_{n+1}} - \sum_{\theta \in \{0,1\}^n \setminus \{0^n\}} Q_{(\theta,1)}$$

$$= \sum_{\substack{\delta \in \{0,1\}^{n+1} \\ \delta_{n+1}=1}} Q'_\delta + \sum_{\substack{v \in r(e_{n+1}) \\ |\sigma(v)|<n+1}} P_v - \sum_{\theta \in \{0,1\}^n \setminus \{0^n\}} \left(Q'_{(\theta,1)} + \sum_{\substack{v \in r(\theta,1) \\ |\sigma(v)|<n+1}} P_v \right)$$

$$= \sum_{\theta \in \{0,1\}^n} Q'_{(\theta,1)} + \sum_{\substack{v \in r(e_{n+1}) \\ |\sigma(v)|<n+1}} P_v - \sum_{\theta \in \{0,1\}^n \setminus \{0^n\}} Q'_{(\theta,1)} - \sum_{\substack{\theta \in \{0,1\}^n \setminus \{0^n\} \\ v \in r(\theta,1) \\ |\sigma(v)|<n+1}} P_v$$

$$= Q'_{(0^n,1)} + \sum_{\substack{v \in r(0^n,1) \\ |\sigma(v)|<n+1}} P_v \qquad\qquad \square$$

Corollary 5.12. *For $\omega \in \{0,1\}^n \setminus \{0^n\}$ with $|r(\omega)| < \infty$, we have*

$$\prod_{\omega_i=1} Q_{e_i} \prod_{\omega_j=0}(1 - Q_{e_j}) = \sum_{v \in r(\omega)} P_v.$$

Proof. Take $\omega \in \{0,1\}^n \setminus \{0^n\}$ with $|r(\omega)| < \infty$. Then $\omega \notin \Delta$. Hence Lemma 4.3 and the definition of $r'(\omega)$ imply that $|\sigma(v)| < |\omega|$ for all $v \in r(\omega)$, and by definition, $Q'_\omega = 0$. Thus the conclusion follows from Lemma 5.11. $\square$

Lemma 5.13. *For $e_n \in \mathcal{G}^1$, we have*

$$S_{e_n} S_{e_n}^* = U_{s(e_n)} \left(\sum_{x \in X(e_n)} t_{(e_n,x)} t_{(e_n,x)}^* \right) U_{s(e_n)}^*.$$

Proof. Lemma 5.3 shows that the U_x, $x \in X(e_n)$ have mutually orthogonal range projections, and the result then follows from the definition of S_{e_n}. $\square$

Lemma 5.14. *For each $v \in G^0$,*

$$\left\{ \sum_{x \in X(e_n)} t_{(e_n,x)} t_{(e_n,x)}^* : n \in \mathbb{N}, s(e_n) = v \right\}$$

is a collection of pairwise orthogonal projections dominated by q_v. Moreover, $G^0_{\mathrm{rg}} \subset E^0_{\mathrm{rg}}$, and if $v \in G^0_{\mathrm{rg}}$ then

$$\sum_{\{n \in \mathbb{N} : s(e_n)=v\}} \left(\sum_{x \in X(e_n)} t_{(e_n,x)} t_{(e_n,x)}^* \right) = q_v.$$

Proof. Proposition 3.14 shows that $G^0_{\mathrm{rg}} \subset E^0_{\mathrm{rg}}$ and both equations then follow from the Cuntz-Krieger relations in $C^*(E)$. $\square$

Proposition 5.15. *The collection $\{P_v : v \in G^0\}$ and $\{S_{e_n} : e_n \in \mathcal{G}^1\}$ is an Exel-Laca $\mathcal{G}$-family in $C^*(E)$.*

Proof. By Lemma 5.6 (1) and Lemma 5.8 (1) and (2), the collection $\{P_v : v \in G^0\}$ and $\{Q_{e_n} : e_n \in \mathcal{G}^1\}$ satisfies the conditions (1), (2), and (3) of Definition 2.2. It follows from [11, Corollary 2.18] that to establish the $\{P_v\}$ and the $\{Q_{e_n}\}$ satisfy Condition (EL), it suffices to verify Condition (4) of Definition 2.2 when $\lambda \cup \mu = \{e_1, \ldots, e_n\}$ for some n, and this follows from Corollary 5.12. The conditions (2) and (3) in Definition 2.3 and the fact that the elements of $\{S_{e_n} : e_n \in \mathcal{G}^1\}$ have mutually orthogonal ranges follow from Lemma 5.13 and Lemma 5.14. $\qquad\square$

Proposition 5.16. *There is a strongly continuous action β of $\mathbb{T}$ on $C^*(E)$ satisfying*

- *$\beta_z(q_x) = q_x$ for $x \in E^0$,*
- *$\beta_z(t_{\overline{x}}) = t_{\overline{x}}$ for $x \in W_+ \sqcup \Gamma_+$, and*
- *$\beta_z(t_{(e_n,x)}) = z t_{(e_n,x)}$ for $e_n \in \mathcal{G}^1, x \in X(e_n)$.*

Moreover, there is an injective homomorphism $\phi \colon C^*(\mathcal{G}) \to C^*(E)$ such that $\phi(p_v) = P_v$ and $\phi(s_e) = S_e$, and ϕ is equivariant for β and the gauge action on $C^*(E)$.

Proof. The existence of β follows from a standard argument using the universal property of $C^*(E)$.

The first statement of [11, Corollary 3.5] implies that $C^*(\mathcal{G})$ is universal for Exel-Laca $\mathcal{G}$-families. Hence there is a homomorphism $\phi : C^*(\mathcal{G}) \to C^*(E)$ such that $\phi(p_v) = P_v$ and $\phi(s_e) = S_e$. To prove that ϕ is injective, we first show that ϕ is equivariant for β and the gauge action on $C^*(E)$, and then apply the gauge-invariant uniqueness theorem for ultragraphs as stated in [11, Corollary 3.5].

It suffices to show that $\beta_z(P_v) = P_v$ and $\beta_z(S_e) = z S_e$ for $v \in G^0$, $e \in \mathcal{G}^1$ and $z \in \mathbb{T}$. Each β_z fixes t_α for every $\alpha \in F^*$, and hence fixes the partial isometries U_x of Definition 5.2. Hence β has the desired properties by definition of the S_e and P_v. $\qquad\square$

The defining properties of the homomorphism $\phi : C^*(\mathcal{G}) \to C^*(E)$ of the preceding proposition imply that

$$\phi(p_{r(e_n)}) = \phi(s_{e_n}^* s_{e_n}) = S_{e_n}^* S_{e_n} = Q_{e_n}$$

for all n, so for all $\omega \in \{0,1\}^n \setminus \{0^n\}$ we have

$$\phi(p_{r(\omega)}) = \phi\Big(\prod_{\omega_i=1} p_{r(e_i)} \prod_{\omega_j=0} (1 - p_{r(e_j)}) \Big) = \prod_{\omega_i=1} Q_{e_i} \prod_{\omega_j=0} (1 - Q_{e_j}) = Q_\omega.$$

The following proposition shows that the sets $r'(\omega)$ of the preceding section and the projections Q'_ω discussed in this section satisfy a similar relationship (also compare Lemma 4.4 with Lemma 5.7, and Remark 4.5 with Lemma 5.9).

Proposition 5.17. *For $\omega \in \bigsqcup_{n=1}^\infty (\{0,1\}^n \setminus \{0^n\})$, the set $r'(\omega)$ is in $\mathcal{G}^0$, and we have $\phi(p_{r'(\omega)}) = Q'_\omega$.*

Proof. The set $r'(\omega)$ belongs to $\mathcal{G}^0$ by the definitions of $r'(\omega)$ and the algebra $\mathcal{G}^0$. That $\phi(p_{r'(\omega)}) = Q'_\omega$ follows from the definition of $r'(\omega)$ and Lemma 5.11. $\qquad\square$

We next determine the image of the injection ϕ of Proposition 5.16.

Lemma 5.18. *For all $x \in E^0$, we have $U_x U_x^* \in \phi(C^*(\mathcal{G}))$.*

Proof. For $x = v \in G^0$, we have $U_v U_v^* = P_v \in \phi(C^*(\mathcal{G}))$. For $x = \omega \in \Delta$, we have

$$U_\omega U_\omega^* = \prod_{\omega_i=1} Q_{e_i} \prod_{\omega_j=0} (1 - Q_{e_j}) - \sum_{\substack{v \in r(\omega) \\ |\sigma(v)|<|\omega|}} P_v \in \phi(C^*(\mathcal{G}))$$

by Lemma 5.11. $\qquad\square$

Lemma 5.19. *Let $\alpha \in E^*$, and suppose $s_E(\alpha) \in W_0 \sqcup \Gamma_0$. Let*

$$\alpha = g_0 \cdot (e_{n_1}, x_1) \cdot g_1 \cdot (e_{n_2}, x_2) \cdot g_2 \cdots (e_{n_k}, x_k) \cdot g_k$$

be the unique expression for α such that each $e_{n_i} \in \mathcal{G}^1$, $x_i \in X(e_{n_i})$, and $g_i \in F^*$ as in Lemma 4.1. Then

$$t_\alpha = S_{e_{n_1}} S_{e_{n_2}} \cdots S_{e_{n_k}} U_{r_E(\alpha)}.$$

Proof. The proof proceeds by induction on k. When $k = 0$, the path $\alpha = g_0$ belongs to F^* with $s_E(\alpha) \in W_0 \sqcup \Gamma_0$. By Definition 4.7, we have $\alpha = f_{r_E(\alpha)}$. Hence $t_\alpha = U_{r_E(\alpha)}$.

Suppose as an inductive hypothesis that the result holds for $k - 1$, and fix

$$\alpha = g_0 \cdot (e_{n_1}, x_1) \cdot g_1 \cdot (e_{n_2}, x_2) \cdot g_2 \cdots (e_{n_k}, x_k) \cdot g_k \in E^*.$$

Let $\alpha' = g_0 \cdot (e_{n_1}, x_1) \cdot g_1 \cdot (e_{n_2}, x_2) \cdot g_2 \cdots (e_{n_{k-1}}, x_{k-1}) \cdot g_{k-1}$. Then $\alpha = \alpha' \cdot (e_{n_k}, x_k) \cdot g_k$. By the inductive hypothesis,

$$t_\alpha = t_{\alpha'} t_{(e_{n_k}, x_k)} t_{g_k} = S_{e_{n_1}} S_{e_{n_2}} \cdots S_{e_{n_{k-1}}} U_{r_E(\alpha')} t_{(e_{n_k}, x_k)} t_{g_k}.$$

The path $f_{x_k} g_k$ satisfies $s_E(f_{x_k} g_k) = s_E(f_{x_k}) \in W_0 \sqcup \Gamma_0$ and $r_E(f_{x_k} g_k) = r_E(g_k) = r_E(\alpha)$. By Definition 4.7 $f_{r_E(\alpha)} = f_{x_k} g_k$. Hence $U_{r_E(\alpha)} = U_{x_k} t_{g_k}$, and Lemma 5.3 implies

$$S_{e_{n_k}} U_{r_E(\alpha)} = \left(U_{s(e_{n_k})} \sum_{x \in X(e_{n_k})} t_{(e_{n_k}, x)} U_x^* \right) (U_{x_k} t_{g_k}) = U_{s(e_{n_k})} t_{(e_{n_k}, x_k)} t_{g_k}.$$

Since $r_E(\alpha') = s_E((e_{n_k}, x_k)) = s(e_{n_k})$,

$$t_\alpha = S_{e_{n_1}} S_{e_{n_2}} \cdots S_{e_{n_{k-1}}} S_{e_{n_k}} U_{r_E(\alpha)}. \qquad\square$$

The sum $\sum_{x \in W_0 \sqcup \Gamma_0} q_x$ converges strictly to a projection $Q \in \mathcal{M}(C^*(E))$ such that

$$Q t_\alpha t_\beta^* = \begin{cases} t_\alpha t_\beta^* & \text{if } s(\alpha) \in W_0 \sqcup \Gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

(see [14, Lemma 2.10] or [22, Lemma 2.1.13] for details). We then have

$$QC^*(E)Q = \overline{\text{span}} \left\{ t_\alpha t_{\alpha'}^* \in C^*(E) : \alpha, \alpha' \in E^* \text{ with } r_E(\alpha) = r_E(\alpha') \right.$$

$$\text{and } s_E(\alpha), s_E(\alpha') \in W_0 \sqcup \Gamma_0\}.$$

Proposition 5.20. *We have $\phi(C^*(\mathcal{G})) = QC^*(E)Q$.*

Proof. Let $x \in E^0$. Since $s_E(f_x) \in W_0 \sqcup \Gamma_0$, we have $U_x = QU_x$. This and the definitions of $\{P_v : v \in G^0\}$ and $\{S_{e_n} : e_n \in \mathcal{G}^1\}$ imply that each P_v and each S_{e_n} is in $QC^*(E)Q$. Hence $\phi(C^*(\mathcal{G})) \subset QC^*(E)Q$. Let $\alpha, \alpha' \in E^*$ such that $r_E(\alpha) = r_E(\alpha') = x \in E^0$ and $s_E(\alpha), s_E(\alpha') \in W_0 \sqcup \Gamma_0$. By Lemma 5.19,

$$t_\alpha t_{\alpha'}^* = S_{e_{n_1}} S_{e_{n_2}} \cdots S_{e_{n_k}} U_x U_x^* S_{e_{m_l}}^* \cdots S_{e_{m_2}}^* S_{e_{m_1}}^*$$

for some $e_{n_i}, e_{m_j} \in \mathcal{G}^1$. Lemma 5.18 therefore implies that $t_\alpha t_{\alpha'}^* \in \phi(C^*(\mathcal{G}))$. Hence $QC^*(E)Q \subset \phi(C^*(\mathcal{G}))$. $\square$

Lemma 5.21. *The projection Q is full.*

Proof. For $x \in E^0$, we have $U_x U_x^* \in QC^*(E)Q$. Hence $q_x = U_x^* U_x$ is in the ideal generated by $QC^*(E)Q$. Since the ideal generated by $\{q_x : x \in E^0\}$ is $C^*(E)$, the ideal generated by $QC^*(E)Q$ is also $C^*(E)$. $\square$

Theorem 5.22. *The homomorphism ϕ of Proposition 5.16 is an isomorphism from $C^*(\mathcal{G})$ to the full corner $QC^*(E)Q$. Consequently $C^*(\mathcal{G})$ and $C^*(E)$ are Morita equivalent.*

Proof. This follows from Proposition 5.16, Proposition 5.20, and Lemma 5.21. $\square$

Theorem 5.23. *The three classes of graph algebras, of Exel-Laca algebras, and of ultragraph algebras coincide up to Morita equivalence.*

Proof. By [20, Theorem 4.5 and Remark 4.6], every Exel-Laca algebra is isomorphic to an ultragraph algebra. Moreover, by [20, Theorem 4.5 and Proposition 6.6], every ultragraph algebra is isomorphic to a full corner of an Exel-Laca algebra.

Proposition 3.1 of [20] implies that every graph C^*-algebra is isomorphic to an ultragraph algebra. Finally, Theorem 5.22 implies that every ultragraph algebra is Morita equivalent to a graph algebra. $\square$

Remark 5.24. Note that Theorem 5.22 also shows how to realize an Exel-Laca algebra as the full corner of a graph algebra. If $\mathcal{A}$ is a countably indexed $\{0, 1\}$-matrix with no zero rows, let $\mathcal{G}_A$ be the ultragraph of [20, Definition 2.5], which has A as its edge matrix. It follows from [20, Theorem 4.5] that the Exel-Laca algebra $\mathcal{O}_A$ is isomorphic to $C^*(\mathcal{G}_A)$. If we let E be a graph constructed from $\mathcal{G}_A$ as in Section 3, then $\mathcal{O}_A$ is isomorphic to a full corner of $C^*(E)$. It is noteworthy that it seems very difficult to see how to construct the graph E directly from the infinite matrix A without at least implicit reference to the ultragraph $\mathcal{G}_A$.

Remark 5.25. With the notation as above, it is straightforward to see that the following conditions are equivalent:

(i) The homomorphism ϕ of Proposition 5.16 is surjective.
(ii) The projection Q is the unit of $\mathcal{M}C^*(E)$.

(iii) $W_+ \sqcup \Gamma_+ = \emptyset$
(iv) $\Delta = \emptyset$.
(v) For all $e \in \mathcal{G}^1$, $|r(e)| < \infty$.

In this case, the graph $E = (E^0, E^1, r_E, s_E)$ is obtained as $E^0 = G^0$, $E^1 = \{(e, x) : e \in \mathcal{G}^1, x \in r(e)\}$, $s_E(e, x) = s(e)$ and $r_E(e, x) = x$. In other words, the graph E is obtained from the ultragraph $\mathcal{G}$ by changing each ultraedge $e \in \mathcal{G}^1$ to a set of (ordinary) edges $\{(e, x) : x \in r(e)\}$.

As a consequence of Theorem 5.22, we obtain the following characterization of real rank zero for ultragraph algebras.

Proposition 5.26. *Let $\mathcal{G}$ be an ultragraph. Then $C^*(\mathcal{G})$ has real rank zero if and only if $\mathcal{G}$ satisfies Condition (K).*

Proof. Let E be a graph constructed from $\mathcal{G}$ as in Section 3. By Theorem 5.22, $C^*(E)$ is Morita equivalent to $C^*(\mathcal{G})$. Hence [3, Theorem 3.8] implies that $C^*(\mathcal{G})$ has real rank zero if and only if $C^*(E)$ has real rank zero. By [10, Theorem 3.5], $C^*(E)$ has real rank zero if and only if E satisfies Condition (K). By Proposition 4.10, E satisfies Condition (K) if and only if $\mathcal{G}$ satisfies Condition (K). $\qquad\square$

6. Gauge-invariant ideals

We continue in this section with a fixed ultragraph $\mathcal{G}$, and let E be a graph constructed from $\mathcal{G}$ as in Section 3. We let $Q \in \mathcal{M}(C^*(E))$ and $\phi : C^*(\mathcal{G}) \to QC^*(E)Q$ be as in Theorem 5.22.

By Theorem 5.22, the homomorphism ϕ induces a bijection from the set of ideals of $C^*(\mathcal{G})$ to the set of ideals of $C^*(E)$. We will show in Proposition 6.10 that this bijection restricts to a bijection between gauge-invariant ideals of $C^*(\mathcal{G})$ and gauge-invariant ideals of $C^*(E)$.

Let β be the action of $\mathbb{T}$ on $C^*(E)$ constructed in the proof of Proposition 5.16. Specifically, $\beta_z(q_x) = q_x$ for $x \in E^0$, $\beta_z(t_{\overline{x}}) = t_{\overline{x}}$ for $x \in W_+ \sqcup \Gamma_+$, and $\beta_z(t_{(e_n, x)}) = z t_{(e_n, x)}$ for $e_n \in \mathcal{G}^1$, $x \in X(e_n)$. Let $\alpha \in E^*$ and let

$$\alpha = g_0 \cdot (n_1, x_1) \cdot g_1 \cdot (n_2, x_2) \cdot g_2 \cdots (n_k, x_k) \cdot g_k$$

be the unique expression for α where each $n_i \in \mathbb{N}$, $x_i \in X(e_{n_i})$ and $g_i \in F^*$ as in Lemma 4.1. We define $m(\alpha) = \max\{n_1, \ldots, n_k\}$ and $l(\alpha) = k$. Then one can verify that $\beta_z(t_\alpha) = z^{l(\alpha)} t_\alpha$. It follows (see, for example, the argument of [14, Corollary 3.3]) that the fixed point algebra $C^*(E)^\beta$ of the action β satisfies

$$C^*(E)^\beta = \overline{\operatorname{span}}\{t_\alpha t_{\alpha'}^* : \alpha, \alpha' \in E^* \text{ with } l(\alpha) = l(\alpha')\}.$$

We define

$$C^*(E)^\circ := \overline{\operatorname{span}}\{t_\alpha t_\alpha^* \in C^*(E) : \alpha \in E^*\} \subset C^*(E)^\beta.$$

Then $C^*(E)^\circ$ is an abelian C^*-subalgebra of $C^*(E)$.

Lemma 6.1. *For $k, n \in \mathbb{N}$ and $x \in E^0$, define*

$$E^*_{k,n,x} := \{\alpha \in E^* : l(\alpha) = k, \ m(\alpha) \leq n, \ r_E(\alpha) = x\}.$$

Then $E^*_{k,n,x}$ is finite and $t^*_\alpha t_{\alpha'} = 0$ for $\alpha, \alpha' \in E^*_{k,n,x}$ with $\alpha \neq \alpha'$. Hence

$$\mathfrak{A}_{k,n,x} := \mathrm{span}\left\{t_\alpha t^*_{\alpha'} : \alpha, \alpha' \in E^*_{k,n,x}\right\}$$

is a C^*-algebra isomorphic to $M_{|E^*_{k,n,x}|}(\mathbb{C})$.

Proof. For each $x \in E^0$, only finitely many paths g in F^* satisfy $r_E(g) = x$. Hence the set $E^*_{k,n,x}$ is finite. It is easy to see that the $t_\alpha t^*_{\alpha'}$ are matrix units. $\square$

Lemma 6.2. *For $k, n \in \mathbb{N}$ and $x, y \in E^0$ with $x \neq y$, we have $\mathfrak{A}_{k,n,x}\mathfrak{A}_{k,n,y} \subset \mathfrak{A}_{k,n,y}$ if there exists a path in F^* from x to y, $\mathfrak{A}_{k,n,x}\mathfrak{A}_{k,n,y} \subset \mathfrak{A}_{k,n,x}$ if there exists a path in F^* from y to x, and $\mathfrak{A}_{k,n,x}\mathfrak{A}_{k,n,y} = 0$ otherwise.*

Proof. We begin by recalling that for any α, α', β, and β' in E^*, the Cuntz-Krieger relations imply that

$$(6.1) \qquad t_\alpha t^*_{\alpha'} t_\beta t^*_{\beta'} = \begin{cases} t_{\alpha\mu} t^*_{\beta'} & \text{if } \beta = \alpha'\mu \text{ for some } \mu \in E^* \\ t_\alpha t^*_{\beta'\nu} & \text{if } \alpha' = \beta\nu \text{ for some } \nu \in E^* \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\alpha' \in E^*_{k,n,x}$ and $\beta \in E^*_{k,n,y}$ satisfy $\beta = \alpha'\mu$ for some $\mu \in E^*$. We claim that μ is a path in F^* from x to y, and that $\alpha\mu \in E^*_{k,n,y}$. Since $r(\alpha') = x$ and $r(\beta) = y$, μ is a path from x to y. Since $l(\alpha') = l(\beta)$, Lemma 4.1 and the definition of l imply that $\mu \in F^*$. Hence $l(\alpha\mu) = l(\alpha) = k$. Since every edge in $\alpha\mu$ is an edge in α or an edge in β, we also have $m(\alpha\mu) \leq \max\{m(\alpha), m(\beta)\} \leq n$. Since $r(\alpha\mu) = r(\mu) = r(\beta) = y$, it follows that $\alpha\mu \in E^*_{k,n,y}$ as claimed.

A symmetric argument now shows that if $\alpha' \in E^*_{k,n,x}$ and $\beta \in E^*_{k,n,y}$ satisfy $\alpha' = \beta\nu$ for some $\nu \in E^*$, then ν is a path in F^* from y to x and $\beta'\nu \in E^*_{k,n,y}$.

By Lemma 4.6, F^* contains no return paths. Since $x \neq y$, it follows that there cannot exist paths $\mu, \nu \in F^*$ such that μ is a path from x to y and ν is a path from y to x. Combining this with the preceding paragraphs and with (6.1) proves the result. $\square$

Lemma 6.3. *Let $\mathfrak{A}_0$ and $\mathfrak{A}'$ be finite-dimensional C^*-subalgebras of a C^*-algebra such that $\mathfrak{A}_0\mathfrak{A}' \subset \mathfrak{A}_0$. Then $\mathfrak{A} := \mathfrak{A}_0 + \mathfrak{A}'$ is a finite-dimensional C^*-algebra whose center is contained in the C^*-algebra generated by the center of $\mathfrak{A}_0$ and the center of $\mathfrak{A}'$.*

Proof. It is clear that $\mathfrak{A} := \mathfrak{A}_0 + \mathfrak{A}'$ is finite dimensional. It is easy to check that $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{A}_0 \subset \mathfrak{A}$ is an ideal. Since $\mathfrak{A}_0$ has a unit p_0, $\mathfrak{A}$ is the direct sum of $\mathfrak{A}_0$ and the C^*-subalgebra $(1 - p_0)\mathfrak{A} \subset \mathfrak{A}$ where 1 is the unit of $\mathfrak{A}$. Since $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{A}'$, the $*$-homomorphism $\mathfrak{A}' \ni x \mapsto (1 - p_0)x \in (1 - p_0)\mathfrak{A}$ is a surjection between finite-dimensional C^*-algebras. Thus its restriction to the center of $\mathfrak{A}'$ is a surjection onto the center of $(1 - p_0)\mathfrak{A}$. This implies that the center of $(1 - p_0)\mathfrak{A}$ is contained in the C^*-algebra generated by the center of $\mathfrak{A}_0$ and the center of $\mathfrak{A}'$ because p_0 is in the center of $\mathfrak{A}_0$. Since

the center of $\mathfrak{A}$ is the direct sum of the center of $\mathfrak{A}_0$ and the center of $(1 - p_0)\mathfrak{A}$, it is contained in the C^*-algebra generated by the center of $\mathfrak{A}_0$ and the center of $\mathfrak{A}'$. $\square$

Lemma 6.4. *Let $k, n \in \mathbb{N}$ and let λ be a finite subset E^0. Then $\mathfrak{A}_{k,n,\lambda} := \sum_{x \in \lambda} \mathfrak{A}_{k,n,x}$ is a finite-dimensional C^*-algebra whose center is contained in $C^*(E)^\circ$.*

Proof. The proof proceeds by induction on $|\lambda|$. When $|\lambda| = 1$, this follows from Lemma 6.1. Suppose the statement holds whenever $|\lambda| = m$. Fix a finite subset $\lambda \subset E^0$ with $|\lambda| = m + 1$. By Lemma 4.6, F^* contains no return paths, so there exists $x_0 \in \lambda$ such that there is no path in F^* from x_0 to any other vertex in λ. Let $\lambda' = \lambda \setminus \{x_0\}$. Then Lemma 6.2 implies that $\mathfrak{A}_{k,n,x_0}\mathfrak{A}_{k,n,\lambda \setminus \{x_0\}} \subset \mathfrak{A}_{k,n,x_0}$. Hence $\mathfrak{A}_{k,n,\lambda}$ is a finite-dimensional C^*-algebra whose center is contained in $C^*(E)^\circ$ by the inductive hypothesis applied to λ', and Lemma 6.3. $\square$

Lemma 6.5. *Let $\lambda_1, \lambda_2, \ldots$ be an increasing sequence of finite subsets of E^0 such that $\bigcup_{n=1}^\infty \lambda_n = E^0$. For $n \in \mathbb{N}$ let $\mathfrak{A}_n := \sum_{k=1}^n \mathfrak{A}_{k,n,\lambda_n}$. Then $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ is an increasing sequence of finite-dimensional C^*-algebras whose centers are contained in $C^*(E)^\circ$, and the union $\bigcup_{n=1}^\infty \mathfrak{A}_n$ is dense in $C^*(E)^\beta$.*

Proof. Equation 6.1 implies that $\mathfrak{A}_{k',n,\lambda_n}\mathfrak{A}_{k,n,\lambda_n} \subset \mathfrak{A}_{k,n,\lambda_n}$ for $k' \leq k$. An argument similar to the proof of Lemma 6.4 therefore shows that $\mathfrak{A}_n$ is a finite-dimensional C^*-algebra whose center is contained in $C^*(E)^\circ$. By definition, $\{\mathfrak{A}_n : n \in \mathbb{N}\}$ is increasing. The union $\bigcup_{n=1}^\infty \mathfrak{A}_n$ is dense in $C^*(E)^\beta$ because it contains all the spanning elements. $\square$

Lemma 6.6. *Every ideal I of $C^*(E)^\beta$ is generated as an ideal by $I \cap C^*(E)^\circ$.*

Proof. Let $\lambda_1, \lambda_2, \ldots$ and $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be as in Lemma 6.5. Then I is generated as an ideal by $\bigcup_{n=1}^\infty I \cap \mathfrak{A}_n$. For each n, the algebra $C^*(E)^\circ$ contains the center of the finite-dimensional C^*-algebra $\mathfrak{A}_n$, so $I \cap \mathfrak{A}_n$ is generated as an ideal by $I \cap \mathfrak{A}_n \cap C^*(E)^\circ$. Hence I is generated as an ideal by $I \cap C^*(E)^\circ$. $\square$

Proposition 6.7. *Let I be an ideal of $C^*(E)$. Then I is β-invariant if and only if I is generated as an ideal by $I \cap C^*(E)^\circ$.*

To prove the proposition, we first present a well-known technical lemma, an exact statement of which we have found difficult to locate in the literature.

Lemma 6.8. *Let A be a C^*-algebra and let β be a strongly continuous action of $\mathbb{T}$ by automorphisms of A. An ideal I of A is β-invariant if and only if it is generated as an ideal by $I \cap A^\beta$.*

Proof. If I is generated as an ideal by $I \cap A^\beta$, then it is clearly β-invariant.

Now suppose that I is β-invariant. Then $I \cap A^\beta = I^\beta$. Moreover, β descends to an action $\widetilde{\beta}$ of $\mathbb{T}$ on A/I, and averaging over β and $\widetilde{\beta}$ gives faithful conditional expectations $\Phi : A \to A^\beta$ and $\widetilde{\Phi} : A/I \to (A/I)^{\widetilde{\beta}}$ such that $\widetilde{\Phi}(a + I) = \Phi(a) + I$.

Let $J \subset I$ be the ideal of A generated by I^β; we must show that $J = I$. Fix $a \in J$. Then $a^*a \in J$, so $\Phi(a^*a) \in J^\beta = I^\beta$. Thus $\widetilde{\Phi}(a^*a + I) = \Phi(a^*a) + I = 0_{A/I}$ since $I^\beta \subset I$. Since $\widetilde{\Phi}$ is faithful, $a^*a + I = 0_{A/I}$, so the C^*-identity implies $a + I = 0_{A/I}$, and $a \in I$. $\square$

Proof of Proposition 6.7. Since elements in $C^*(E)^\circ$ are fixed by β, if I is generated by $I \cap C^*(E)^\circ$, then I is β-invariant. Conversely suppose that I is β-invariant. Then Lemma 6.8 shows that I is generated as an ideal of $C^*(E)$ by $I \cap C^*(E)^\beta$, and Lemma 6.6 implies that $I \cap C^*(E)^\beta$ is generated as an ideal of $C^*(E)^\beta$ by $I \cap C^*(E)^\circ$. $\qquad\square$

The following proposition holds for a general graph E.

Proposition 6.9. *An ideal I of $C^*(E)$ is gauge invariant if and only if I is generated by $I \cap C^*(E)^\circ$.*

Proof. Since $C^*(E)^\circ$ is in the fixed point algebra of the gauge action, the ideal generated by a C^*-subalgebra of $C^*(E)^\circ$ is gauge invariant. Conversely, let I be a gauge-invariant ideal of $C^*(E)$ and J be the ideal generated by $I \cap C^*(E)^\circ$. Since $I \cap C^*(E)^\circ \subset J \subset I$, we have $J \cap C^*(E)^\circ = I \cap C^*(E)^\circ$. Theorem 3.6 of [1] implies that each gauge-invariant ideal of $C^*(E)$ is uniquely determined by its intersection with $C^*(E)^\circ$. Since both I and J are gauge invariant, it follows that $I = J$. $\qquad\square$

Proposition 6.10. *Let I be an ideal of $C^*(\mathcal{G})$. Then I is invariant under the gauge action on $C^*(\mathcal{G})$ if and only if the ideal generated by $\phi(I)$ is invariant under the gauge action on $C^*(E)$.*

Proof. Let J be the ideal generated by $\phi(I)$ in $C^*(E)$. By Proposition 5.16, I is invariant under γ if and only if J is invariant under β. The latter condition is equivalent to the gauge invariance of J by Proposition 6.7 and Proposition 6.9. $\qquad\square$

7. Quotients by gauge-invariant ideals

In this section, give a more explicit description of the bijection between gauge-invariant ideals of $C^*(\mathcal{G})$ and gauge-invariant ideals of $C^*(E)$ stated in Proposition 6.10. To do this we use the classifications of gauge-invariant ideals in graph algebras [1, Theorem 3.6] and in ultragraph algebras [11, Theorem 6.12]. We also describe quotients of ultragraph algebras by gauge-invariant ideals as full corners in graph algebras.

First recall from [11, Section 6] that an admissible pair for $\mathcal{G}$ consists of a subset $\mathcal{H}$ of $\mathcal{G}^0$ and a subset V of G^0 such that:

- $\mathcal{H}$ is an ideal: if $U_1, U_2 \in \mathcal{H}$ then $U_1 \cup U_2 \in \mathcal{H}$, and if $U_1 \in \mathcal{G}^0$, $U_2 \in \mathcal{H}$ and $U_1 \subset U_2$, then $U_1 \in \mathcal{H}$;
- $\mathcal{H}$ is hereditary: if $e \in \mathcal{G}^1$ and $\{s(e)\} \in \mathcal{H}$, then $r(e) \in \mathcal{H}$;
- $\mathcal{H}$ is saturated: if $v \in G^0_{\mathrm{rg}}$ and $r(e) \in \mathcal{H}$ for all $e \in s^{-1}(v)$, then $\{v\} \in \mathcal{H}$; and
- $V \subset \mathcal{H}^{\mathrm{fin}}_\infty$, where

$$\mathcal{H}^{\mathrm{fin}}_\infty := \{v \in G^0 : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap \{e \in \mathcal{G}^1 : r(e) \notin \mathcal{H}\}| < \infty\}.$$

Theorem 6.12 of [11] shows that there is a bijection $I \mapsto (\mathcal{H}_I, V_I)$ between gauge-invariant ideals of $C^*(\mathcal{G})$ and admissible pairs for $\mathcal{G}$. Specifically, $\mathcal{H}_I = \{U \in \mathcal{G}^0 : p_U \in I\}$ and $V_I = \{v \in (\mathcal{H}_I)^{\mathrm{fin}}_\infty : p_v - \sum_{e \in s^{-1}(v), r(e) \notin \mathcal{H}_I} s_e s_e^* \in I\}$.

We must also recall from [2] some terminology for a directed graph $E = (E^0, E^1, r_E, s_E)$. A subset H of E^0 is said to be *hereditary* if $r_E(\alpha) \in H$ whenever $\alpha \in E^1$ and $s_E(\alpha) \in H$. A hereditary subset H is said to be *saturated* if $x \in H$ whenever $x \in E^0_{\mathrm{rg}}$ and $r_E(\alpha) \in H$ for all $\alpha \in s_E^{-1}(x)$. If $H \subset E^0$ is saturated hereditary, then we define

$$H^{\mathrm{fin}}_\infty := \{v \in E^0 : |s_E^{-1}(v)| = \infty \text{ and } 0 < |s_E^{-1}(v) \cap r_E^{-1}(E^0 \setminus H)| < \infty\}.$$

Theorem 3.6 of [1] shows that there is a bijection $J \mapsto (H_J, B_J)$ between gauge-invariant ideals of $C^*(E)$ and pairs (H, B) such that $H \subset E^0$ is saturated hereditary, and $B \subset H^{\mathrm{fin}}_\infty$. Specifically,

$$H_J = \{x \in E^0 : q_x \in J\} \quad \text{and} \quad B_J = \{v \in (H_J)^{\mathrm{fin}}_\infty : q_v - \sum_{\alpha \in s_E^{-1}(v), r_E(\alpha) \notin H_J} t_\alpha t_\alpha^* \in J\}.$$

Definition 7.1. For a saturated hereditary ideal $\mathcal{H} \subset \mathcal{G}^0$, we define $\theta(\mathcal{H}) \subset E^0$ by

$$\theta(\mathcal{H}) := \{v \in G^0 : \{v\} \in \mathcal{H}\} \cup \{\omega \in \Delta : r'(\omega) \in \mathcal{H}\}.$$

Proposition 7.2. *If I is a gauge-invariant ideal of $C^*(\mathcal{G})$, and J is the ideal of $C^*(E)$ generated by $\phi(I)$, then $H_J = \theta(\mathcal{H}_I)$, $(H_J)^{\mathrm{fin}}_\infty = (\mathcal{H}_I)^{\mathrm{fin}}_\infty$, and $B_J = V_I$.*

Proof. We use the notation established in Section 3 and Section 5. Let $x \in E^0$. Since $q_x = U_x^* U_x$, we have

$$x \in H_J \iff q_x \in J \iff U_x \in J \iff U_x U_x^* \in J.$$

For $v \in G^0$, we have $U_v U_v^* = \phi(p_v)$. Hence

$$U_v U_v^* \in J \iff p_v \in I \iff \{v\} \in \mathcal{H}_I.$$

Thus $v \in H_J$ if and only if $\{v\} \in \mathcal{H}_I$. Similarly, for $\omega \in \Delta$, we have $Q'_\omega = U_\omega U_\omega^*$ by Definition 5.5, and Proposition 5.17 implies that $\phi(p_{r'(\omega)}) = Q'_\omega$, so

$$U_\omega U_\omega^* \in J \iff p_{r'(\omega)} \in I \iff r'(\omega) \in \mathcal{H}_I.$$

Thus $\omega \in H_J$ if and only if $r'(\omega) \in \mathcal{H}_I$. This shows that $H_J = \theta(\mathcal{H}_I)$.

Next, we show $(H_J)^{\mathrm{fin}}_\infty = (\mathcal{H}_I)^{\mathrm{fin}}_\infty$. Since each $\omega \in \Delta$ satisfies $|s_E^{-1}(\omega)| < \infty$, we have $(H_J)^{\mathrm{fin}}_\infty \subset G^0$. Fix $v \in G^0$. We have $s_E^{-1}(v) = \bigsqcup_{s(e_n) = v} \{(e_n, x) : x \in X(e_n)\}$. Since each $X(e_n)$ is finite, $|s_E^{-1}(v)| = \infty$ if and only if $|s^{-1}(v)| = \infty$. Lemma 4.4 and the conclusion of the preceding paragraph imply that $r(e_n) \in \mathcal{H}_I$ if and only if $X(e_n) \subset H_J$. Hence

$$(7.1) \quad 0 < |s_E^{-1}(v) \cap r_E^{-1}(E^0 \setminus H_J)| < \infty \iff 0 < |s^{-1}(v) \cap \{e \in \mathcal{G}^1 : r(e) \notin \mathcal{H}_I\}| < \infty$$

Thus $(H_J)^{\mathrm{fin}}_\infty = (\mathcal{H}_I)^{\mathrm{fin}}_\infty$.

Finally we show $B_J = V_I$. Fix $v \in (H_J)^{\mathrm{fin}}_\infty = (\mathcal{H}_I)^{\mathrm{fin}}_\infty$. Let $L := \{n : s(e_n) = v, r(e_n) \notin \mathcal{H}_I\}$. By (7.1), we have

$$\{\alpha \in s_E^{-1}(v) : r_E(\alpha) \notin H_J\} = \{(e_n, x) : n \in L, x \in X(e_n) \setminus H_J\}.$$

For $n \in L$ and $x \in X(e_n) \cap H_J$, we have $t_{(e_n,x)}^* t_{(e_n,x)} = q_x \in J$, and hence $t_{(e_n,x)} t_{(e_n,x)}^* \in J$. Thus

$$q_v - \sum_{\alpha \in s_E^{-1}(v), r_E(\alpha) \notin H_J} t_\alpha t_\alpha^* = q_v - \sum_{\substack{n \in L \\ x \in X(e_n) \setminus H_J}} t_{(e_n,x)} t_{(e_n,x)}^*$$

$$= q_v - \sum_{\substack{n \in L \\ x \in X(e_n)}} t_{(e_n,x)} t_{(e_n,x)}^* + \sum_{\substack{n \in L \\ x \in X(e_n) \cap H_J}} t_{(e_n,x)} t_{(e_n,x)}^*$$

belongs to J if and only if

(7.2) $$q_v - \sum_{n \in L, x \in X(e_n)} t_{(e_n,x)} t_{(e_n,x)}^* \in J.$$

Moreover, (7.2) holds if and only if $p_v - \sum_{n \in L} s_{e_n} s_{e_n}^* \in I$ because

$$\phi\Big(p_v - \sum_{n \in L} s_{e_n} s_{e_n}^*\Big) = P_v - \sum_{n \in L} S_{e_n} S_{e_n}^* = U_v\Big(q_v - \sum_{n \in L, x \in X(e_n)} t_{(e_n,x)} t_{(e_n,x)}^*\Big) U_v^*$$

by Lemma 5.13. Hence $B_J = V_I$. $\qquad\square$

Corollary 7.3. *Let I be a gauge-invariant ideal of $C^*(\mathcal{G})$. Then the isomorphism $\phi : C^*(\mathcal{G}) \to QC^*(E)Q$ restricts to an isomorphism of I onto QJQ, where J is the unique gauge-invariant ideal of $C^*(E)$ such that $H_J = \theta(\mathcal{H}_I)$ and $B_J = V_I$.*

Proof. We have $\phi(I) = QJQ$ where J is the ideal of $C^*(E)$ generated by $\phi(I)$. By Proposition 6.10 and Proposition 7.2, J is the gauge-invariant ideal of $C^*(E)$ such that $H_J = \theta(\mathcal{H}_I)$ and $B_J = V_I$. $\qquad\square$

Using Proposition 7.2 and the results of [1], we may now describe quotients of ultragraph algebras by gauge-invariant ideals as full corners in graph algebras.

Definition 7.4. Let I be a gauge-invariant ideal of $C^*(\mathcal{G})$, and let $\mathcal{H}_I$, V_I, and $\theta(\mathcal{H}_I)$ be as above. We define a directed graph $E_I = (E_I^0, E_I^1, r_{E_I}, s_{E_I})$ as follows. The vertex and edge sets are defined by

$$E_I^0 := (E^0 \setminus \theta(\mathcal{H}_I)) \sqcup \{\widetilde{x} : x \in (\mathcal{H}_I)_\infty^{\text{fin}} \setminus V_I\}, \text{ and}$$
$$E_I^1 := r_E^{-1}(E^0 \setminus \theta(\mathcal{H}_I)) \sqcup \{\widetilde{\alpha} : \alpha \in r_E^{-1}((\mathcal{H}_I)_\infty^{\text{fin}} \setminus V_I) \subset E^1\}.$$

The range and source of $e \in r_E^{-1}(E^0 \setminus \theta(\mathcal{H}_I))$ in E_I are the same as those in E. For $\alpha \in r_E^{-1}((\mathcal{H}_I)_\infty^{\text{fin}} \setminus V_I)$ we define $s_{E_I}(\widetilde{\alpha}) := s_E(\alpha)$, and $r_{E_I}(\widetilde{\alpha}) := \widetilde{x}$ where $x = r_E(\alpha) \in (\mathcal{H}_I)_\infty^{\text{fin}} \setminus V_I$.

Corollary 7.5. *With the notation above, $C^*(\mathcal{G})/I$ is isomorphic to a full corner of $C^*(E_I)$.*

Proof. Let J be the ideal of $C^*(E)$ generated by $\phi(I)$. By Theorem 5.22 the homomorphism ϕ induces an isomorphism $\phi_I \colon C^*(\mathcal{G})/I \to \overline{Q}(C^*(E)/J)\overline{Q}$ where $\overline{Q} \in \mathcal{M}(C^*(E)/J)$ is the image of $Q \in \mathcal{M}(C^*(E))$ under the extension of the quotient map to multiplier algebras (see [16, Corollary 2.51]). In particular, the projection $\overline{Q}$ is full. By Proposition 7.2 we obtain $H_J = \theta(\mathcal{H}_I)$ and $(H_J)_\infty^{\mathrm{fin}} \setminus B_J = (\mathcal{H}_I)_\infty^{\mathrm{fin}} \setminus V_I$. By [1, Corollary 3.5] there is an isomorphism $\psi \colon C^*(E)/J \to C^*(E_I)$. Let $Q_I \in \mathcal{M}(C^*(E_I))$ be the image of $\overline{Q}$ under ψ. Then Q_I is full, and $\psi \circ \phi_I \colon C^*(\mathcal{G})/I \to Q_I C^*(E_I) Q_I$ is the desired isomorphism. $\qquad\square$

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